

A New Modified Fifth-Order Implicit Block Method for Solving Stiff and Oscillatory Problems

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Abstract

A new fifth-order modified implicit 3-point super class of block backward differentiation formula for solving oscillatory problems and systems of stiff ordinary differential equations (ODEs) is developed. The method is order five. The stability and convergence properties of the method demonstrate that the method is consistent, zero-stable and almost A-stable which are necessary and sufficient conditions to integrate the stiff systems and oscillatory ODEs. Numerical experiments reveal that the proposed method delivers superior performance compared to existing conventional 3-point Block Backward Differentiation Formula (3BBDF) and 3-point Super Class of Block Backward Differentiation Formula (3SBDF) methods, making it a robust and efficient tool for addressing stiff problems.

1. Introduction

Stiff ordinary differential equations (ODEs) are prevalent in various scientific and engineering disciplines, characterized by solutions exhibiting rapid variations alongside more gradual changes. This disparity in solution behavior poses significant challenges for numerical integration methods, necessitating specialized techniques to ensure both stability and accuracy [1]. Thus, in this paper, we shall be concerned with the numerical solution of the first order stiff ODEs, which often arise in problems involving chemical reactions, heat transfer, electrical circuits, population dynamics, and control systems. These equations are typically characterized by the presence of rapidly decaying components that force explicit methods to take impractically small steps to maintain stability. Hence, most robust implicit or block methods are required to achieve accurate and stable solutions efficiently. Therefore, our focus is on the development of a reliable numerical scheme suitable for such stiff problems of the form:

$$y' = f(x, y), \quad y(a) = y_0, \quad a \leq x \leq b. \quad (1)$$

Traditional explicit methods often fail to efficiently handle stiff ODEs due to stringent stability constraints, leading to impractically small step sizes. In real-world applications, stiff and oscillatory problems commonly occur in chemical reaction kinetics, control systems, electrical circuit simulation, celestial mechanics, and fluid dynamics, where rapid and slow processes coexist. To overcome the numerical difficulties posed by such systems, implicit methods, particularly the Backward Differentiation Formulas (BDF), Extended BDF, and the Super Class of Implicit Extended BDF [2, 3, 4], have been extensively employed to address stiffness, owing to

their favorable stability properties. However, conventional BDF methods can be computationally expensive, especially when applied to large systems, due to the necessity of solving nonlinear equations at each step.

To mitigate these computational challenges, researchers have developed block methods that compute solutions at multiple points simultaneously. Notably, the Block Backward Differentiation Formulas (BBDF), Super class of BBDF and Block Extended BDF and Off-step points block methods have been proposed and analyzed for their effectiveness in solving stiff ODEs [5, 6, 7, 8, 9, 10, 11, 12].

Despite these advancements, there remains a need for further research into the development of highly stable and efficient numerical solvers for stiff IVPs. The exploration of those novel integration schemes continues to be an active area of study. This paper aims to enhance the existing fifth order fully implicit 3-point Super Class of Block Backward Differentiation Formula (BDF) method, initially proposed by [7]. The modification involves replacing the term $-\rho f_{n+k-1}$ with ρf_{n+k-1} , which is expected to improve the accuracy and stability of the method.

2. Mathematical Formulation of the Method

Following the approach of Musa et al. [9], this section derives the proposed method based on the following modification:

$$\sum_{j=0}^5 \alpha_{j,i} y_{n+j-2} = h\beta_{k,i} (f_{n+k} + \rho f_{n+k-1}), \quad k = i = 1, 2, 3. \tag{2}$$

To derive the formula in (2) using Taylor series, the general linear operator L_i associated with equation (2) is defined as:

$$L_i[y(x_n), h]: \alpha_{0,i} y_{n-2} + \alpha_{1,i} y_{n-1} + \alpha_{2,i} y_n + \alpha_{3,i} y_{n+1} + \alpha_{4,i} y_{n+2} + \alpha_{5,i} y_{n+3} - h\beta_{k,i} (f_{n+k} + \rho f_{n+k-1}) = 0, \text{ where } k = i = 1, 2, 3. \tag{3}$$

To derive the first point, we substitute $k = i = 1$ in (3), to obtain the following linear operator L_1 which is given by:

$$L_1[y(x_n), h]: \alpha_{0,1} y_{n-2} + \alpha_{1,1} y_{n-1} + \alpha_{2,1} y_n + \alpha_{3,1} y_{n+1} + \alpha_{4,1} y_{n+2} + \alpha_{5,1} y_{n+3} - h\beta_{1,1} (f_{n+1} + \rho f_n) = 0. \tag{4}$$

The corresponding approximate relationship for equation (4) can be expressed as follows:

$$\alpha_{0,1} y(x_n - 2h) + \alpha_{1,1} y(x_n - h) + \alpha_{2,1} y(x_n) + \alpha_{3,1} y(x_n + h) + \alpha_{4,1} y(x_n + 2h) + \alpha_{5,1} y(x_n + 3h) - h\beta_{1,1} (f(x_n + h) + \rho f(x_n)) = 0. \tag{5}$$

The Taylor series expansion around the point x_n results in the following system of simultaneous linear equations:

$$\begin{aligned} C_{0,1} &= \alpha_{0,1} + \alpha_{1,1} + \alpha_{2,1} + \alpha_{3,1} + \alpha_{4,1} + \alpha_{5,1} = 0, \\ C_{1,1} &= -2\alpha_{0,1} - \alpha_{1,1} + \alpha_{3,1} + 2\alpha_{4,1} + 3\alpha_{5,1} - \beta_{1,1}(1 + \rho) = 0, \\ C_{2,1} &= 2\alpha_{0,1} + \frac{1}{2}\alpha_{1,1} + \frac{1}{2}\alpha_{3,1} + 2\alpha_{4,1} + \frac{9}{2}\alpha_{5,1} - \beta_{1,1} = 0, \\ C_{3,1} &= -\frac{4}{3}\alpha_{0,1} - \frac{1}{6}\alpha_{1,1} + \frac{1}{6}\alpha_{3,1} + \frac{4}{3}\alpha_{4,1} + \frac{9}{2}\alpha_{5,1} - \frac{1}{2}\beta_{1,1} = 0, \\ C_{4,1} &= \frac{2}{3}\alpha_{0,1} + \frac{1}{24}\alpha_{1,1} + \frac{1}{24}\alpha_{3,1} + \frac{2}{3}\alpha_{4,1} + \frac{27}{8}\alpha_{5,1} - \frac{1}{6}\beta_{1,1} = 0, \\ C_{5,1} &= -\frac{4}{15}\alpha_{0,1} - \frac{1}{120}\alpha_{1,1} + \frac{1}{120}\alpha_{3,1} + \frac{4}{15}\alpha_{4,1} + \frac{81}{40}\alpha_{5,1} - \frac{1}{24}\beta_{1,1} = 0. \end{aligned} \tag{6}$$

After normalizing the coefficient of y_{n+1} to 1 and solving the simultaneous equations in (6), we obtain the coefficient for the first point as:

Table 1 Coefficients of the first point formula

$\alpha_{0,1}$	$\alpha_{1,1}$	$\alpha_{2,1}$	$\alpha_{3,1}$	$\alpha_{4,1}$	$\alpha_{5,1}$	$\beta_{1,1}$
$\frac{1}{20} \frac{3\rho - 2}{3\rho + 1}$	$-\frac{3}{4} \frac{2\rho - 1}{3\rho + 1}$	$-\frac{\rho + 3}{3\rho + 1}$	1	$-\frac{3}{4} \frac{\rho - 2}{3\rho + 1}$	$\frac{1}{20} \frac{2\rho - 3}{3\rho + 1}$	$\frac{3}{3\rho + 1}$

Therefore, by putting the obtained values of $\alpha_{j,i}$ and $\beta_{k,i}$ in (3), the first point formula is obtained as:

$$y_{n+1} = -\frac{1}{20} \frac{3\rho - 2}{3\rho + 1} y_{n-2} + \frac{3}{4} \frac{2\rho - 1}{3\rho + 1} y_{n-1} + \frac{\rho + 3}{3\rho + 1} y_n + \frac{3}{4} \frac{\rho - 2}{3\rho + 1} y_{n+2} - \frac{1}{20} \frac{2\rho - 3}{3\rho + 1} y_{n+3} + \frac{3h}{3\rho + 1} f_{n+1} + \frac{3\rho h}{3\rho + 1} f_n. \tag{7}$$

To derive the equation for the second and third point, same procedure is applied as in the derivation of the first point, leading to:

$$y_{n+2} = \frac{1}{5} \frac{2\rho - 3}{6\rho + 13} y_{n-2} - \frac{3\rho - 4}{6\rho + 13} y_{n-1} + \frac{12(\rho - 1)}{6\rho + 13} y_n - \frac{4(\rho - 6)}{6\rho + 13} y_{n+1} + \frac{3}{5} \frac{\rho - 4}{6\rho + 13} y_{n+3} + \frac{12h}{6\rho + 13} f_{n+2} + \frac{12\rho h}{6\rho + 13} f_{n+1},$$

$$y_{n+3} = -\frac{3(\rho - 4)}{12\rho + 137} y_{n-2} + \frac{5(4\rho - 15)}{12\rho + 137} y_{n-1} - \frac{20(3\rho - 10)}{12\rho + 137} y_n + \frac{60(2\rho - 5)}{12\rho + 137} y_{n+1} - \frac{5(13\rho - 60)}{12\rho + 137} y_{n+2} + \frac{60h}{12\rho + 137} f_{n+3} + \frac{60\rho h}{12\rho + 137} f_{n+2}. \tag{8}$$

In order to ensure the absolute stability of the method, we select a value for ρ within the range $(-1,1)$. Specifically, we set $\rho = \frac{1}{10}$ and substitute this value in (7) and (8). The new modified fifth-order implicit block method for solving stiff first-order ordinary differential equations (I3SBDF) is presented as follows:

$$y_{n+1} = \frac{17}{260} y_{n-2} - \frac{6}{13} y_{n-1} + \frac{31}{13} y_n - \frac{57}{52} y_{n+2} + \frac{7}{65} y_{n+3} + \frac{30}{13} h f_{n+1} + \frac{3}{13} h f_n,$$

$$y_{n+2} = -\frac{7}{170} y_{n-2} + \frac{37}{136} y_{n-1} - \frac{27}{34} y_n + \frac{59}{34} y_{n+1} - \frac{117}{680} y_{n+3} + \frac{15}{17} h f_{n+2} + \frac{3}{34} h f_{n+1}, \tag{9}$$

$$y_{n+3} = \frac{117}{1382} y_{n-2} - \frac{365}{691} y_{n-1} + \frac{970}{691} y_n - \frac{1440}{691} y_{n+1} + \frac{2935}{1382} y_{n+2} + \frac{300}{691} h f_{n+3} + \frac{30}{691} h f_{n+2}.$$

3. Order, Error Constant and Consistency of the Method

To determine the order of method corresponding to the equation (9), we rewrite the equation in (9) as:

$$-\frac{17}{260} y_{n-2} + \frac{6}{13} y_{n-1} - \frac{31}{13} y_n + y_{n+1} + \frac{57}{52} y_{n+2} - \frac{7}{65} y_{n+3} = \frac{30}{13} h f_{n+1} + \frac{3}{13} h f_n,$$

$$\frac{7}{170} y_{n-2} - \frac{37}{136} y_{n-1} + \frac{27}{34} y_n - \frac{59}{34} y_{n+1} + y_{n+2} + \frac{117}{680} y_{n+3} = \frac{15}{17} h f_{n+2} + \frac{3}{34} h f_{n+1}, \tag{10}$$

$$-\frac{117}{1382} y_{n-2} + \frac{365}{691} y_{n-1} - \frac{970}{691} y_n + \frac{1440}{691} y_{n+1} - \frac{2935}{1382} y_{n+2} + y_{n+3} = \frac{300}{691} h f_{n+3} + \frac{30}{691} h f_{n+2}.$$

The generalized matrix representation of (10) is defined and expressed as:

$$\sum_{j=0}^1 A_j^* Y_{m+j-1} = h \sum_{j=1}^3 B_j^* F_{m+j-1}, \tag{11}$$

where A_0^* , A_1^* , B_0^* and B_1^* are three by three square matrices defined by:

$$A_0^* = \begin{bmatrix} -\frac{17}{260} & \frac{6}{13} & -\frac{31}{13} \\ \frac{7}{170} & -\frac{37}{136} & \frac{27}{34} \\ -\frac{117}{1382} & \frac{365}{691} & -\frac{970}{691} \end{bmatrix}, A_1^* = \begin{bmatrix} 1 & \frac{57}{52} & -\frac{7}{65} \\ -\frac{59}{34} & 1 & \frac{117}{680} \\ \frac{1440}{691} & -\frac{2935}{1382} & 1 \end{bmatrix}, B_0^* = \begin{bmatrix} 0 & 0 & \frac{3}{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$B_1^* = \begin{bmatrix} \frac{30}{13} & 0 & 0 \\ \frac{3}{34} & \frac{15}{17} & 0 \\ 0 & \frac{300}{691} & \frac{30}{691} \end{bmatrix},$$

and the column vectors Y_{m-1}, Y_m, F_{m-1} and F_m are defined by

$$Y_m = \begin{bmatrix} Y_{n+1} \\ Y_{n+2} \\ Y_{n+3} \end{bmatrix}, Y_{m-1} = \begin{bmatrix} Y_{n-2} \\ Y_{n-1} \\ Y_n \end{bmatrix}, F_m = \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix} \text{ and } F_{m-1} = \begin{bmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix}.$$

Equation (11) can be written as:

$$\begin{bmatrix} -\frac{17}{260} & \frac{6}{13} & -\frac{31}{13} \\ \frac{7}{170} & -\frac{136}{365} & \frac{34}{970} \\ -\frac{117}{1382} & \frac{691}{691} & -\frac{691}{691} \end{bmatrix} \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} + \begin{bmatrix} 1 & \frac{57}{52} & -\frac{7}{65} \\ -\frac{59}{34} & 1 & \frac{117}{680} \\ \frac{1440}{691} & -\frac{2935}{1382} & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{3}{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n-2} \\ f_{n-1} \\ y_n \end{bmatrix} + \begin{bmatrix} \frac{30}{13} & 0 & 0 \\ \frac{3}{34} & \frac{15}{17} & 0 \\ 0 & \frac{30}{691} & \frac{300}{691} \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix}. \tag{12}$$

Let A_0^*, A_1^*, B_0^* and B_1^* be the matrices defined by:

$$A_0^* = (A_0 \ A_1 \ A_2), A_1^* = (A_3 \ A_4 \ A_5), B_0^* = (B_0 \ B_1 \ B_2) \text{ and } B_1^* = (B_3 \ B_4 \ B_5),$$

Where

$$A_0 = \begin{bmatrix} -\frac{17}{260} \\ \frac{7}{170} \\ -\frac{117}{1382} \end{bmatrix}, A_1 = \begin{bmatrix} \frac{6}{13} \\ -\frac{136}{365} \\ \frac{691}{691} \end{bmatrix}, A_2 = \begin{bmatrix} -\frac{31}{13} \\ \frac{34}{970} \\ -\frac{691}{691} \end{bmatrix}, A_3 = \begin{bmatrix} 1 \\ -\frac{59}{34} \\ \frac{1440}{691} \end{bmatrix}, A_4 = \begin{bmatrix} \frac{57}{52} \\ 1 \\ -\frac{2935}{1382} \end{bmatrix}, A_5 = \begin{bmatrix} -\frac{7}{65} \\ \frac{117}{680} \\ 1 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} \frac{3}{13} \\ 0 \\ 0 \end{bmatrix}, B_3 = \begin{bmatrix} \frac{30}{13} \\ \frac{3}{34} \\ 0 \end{bmatrix}, B_4 = \begin{bmatrix} \frac{15}{17} \\ \frac{30}{691} \\ 0 \end{bmatrix}, B_5 = \begin{bmatrix} 0 \\ \frac{300}{691} \\ 0 \end{bmatrix}.$$

Definition 1: The order of the block method (9) and its associated linear difference operator are given by:

$$L(y(x); h) = \sum_{j=0}^5 A_j y(x + jh) - h \sum_{j=0}^5 B_j y'(x + jh). \tag{13}$$

The unique integer p such that $C_q = 0, q = 0(1)p$ and $C_{p+1} \neq 0$, where the C_q are constant (column) matrices are defined by:

$$c_0 = A_0 + A_1 + A_2 + \dots + A_k,$$

$$c_1 = A_1 + 2A_2 + \dots + kA_k - (B_0 + B_1 + B_2 + \dots + B_{k+1}),$$

$$c_q = \frac{1}{q!} (A_1 + 2^q A_2 + \dots + k^q A_k) - \frac{1}{(q-1)!} (B_1 + 2^{q-1} B_2 + \dots + (k+1)^{q-1} B_{k+1}). \tag{14}$$

For $q = 0(1)5$, we have

$$C_0 = \sum_{j=0}^5 A_j = A_0 + A_1 + A_2 + A_3 + A_4 + A_5,$$

$$= \begin{bmatrix} -\frac{17}{260} \\ \frac{7}{170} \\ \frac{117}{1382} \end{bmatrix} + \begin{bmatrix} \frac{6}{13} \\ -\frac{37}{136} \\ \frac{365}{691} \end{bmatrix} + \begin{bmatrix} -\frac{31}{13} \\ \frac{27}{34} \\ -\frac{970}{691} \end{bmatrix} + \begin{bmatrix} \frac{1}{59} \\ -\frac{34}{1440} \\ \frac{691}{691} \end{bmatrix} + \begin{bmatrix} \frac{57}{52} \\ \frac{1}{2935} \\ -\frac{1382}{1382} \end{bmatrix} + \begin{bmatrix} -\frac{7}{65} \\ \frac{117}{680} \\ \frac{1}{1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$C_1 = \sum_{j=0}^5 jA_j - \sum_{j=0}^5 B_j$$

$$= ((0)A_0 + (1)A_1 + (2)A_2 + (3)A_3 + (4)A_4 + (5)A_5) - (B_0 + B_1 + B_2 + B_3 + B_4 + B_5),$$

$$= \left[0 \cdot \begin{bmatrix} -\frac{17}{260} \\ \frac{7}{170} \\ \frac{117}{1382} \end{bmatrix} + 1 \cdot \begin{bmatrix} \frac{6}{13} \\ -\frac{37}{136} \\ \frac{365}{691} \end{bmatrix} + 2 \cdot \begin{bmatrix} -\frac{31}{13} \\ \frac{27}{34} \\ -\frac{970}{691} \end{bmatrix} + 3 \cdot \begin{bmatrix} \frac{1}{59} \\ -\frac{34}{1440} \\ \frac{691}{691} \end{bmatrix} + 4 \cdot \begin{bmatrix} \frac{57}{52} \\ \frac{1}{2935} \\ -\frac{1382}{1382} \end{bmatrix} + 5 \cdot \begin{bmatrix} -\frac{7}{65} \\ \frac{117}{680} \\ \frac{1}{1} \end{bmatrix} \right] - \left[\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{3}{13} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{30}{13} \\ \frac{3}{34} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{15}{17} \\ \frac{30}{691} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{300}{691} \end{bmatrix} \right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$C_2 = \sum_{j=0}^5 \frac{j^2}{2!} A_j - \frac{1}{1!} \sum_{j=0}^5 jB_j,$$

$$= \frac{1}{2!} ((0)^2 A_0 + (1)^2 A_1 + (2)^2 A_2 + (3)^2 A_3 + (4)^2 A_4 + (5)^2 A_5) - \frac{1}{1!} ((0)^1 B_0 + (1)^1 B_1 + (2)^1 B_2 + (3)^1 B_3 + (4)^1 B_4 + (5)^1 B_5),$$

$$= \frac{1}{2!} \left[0^2 \cdot \begin{bmatrix} -\frac{17}{260} \\ \frac{7}{170} \\ \frac{117}{1382} \end{bmatrix} + 1^2 \cdot \begin{bmatrix} \frac{6}{13} \\ -\frac{37}{136} \\ \frac{365}{691} \end{bmatrix} + 2^2 \cdot \begin{bmatrix} -\frac{31}{13} \\ \frac{27}{34} \\ -\frac{970}{691} \end{bmatrix} + 3^2 \cdot \begin{bmatrix} \frac{1}{59} \\ -\frac{34}{1440} \\ \frac{691}{691} \end{bmatrix} + 4^2 \cdot \begin{bmatrix} \frac{57}{52} \\ \frac{1}{2935} \\ -\frac{1382}{1382} \end{bmatrix} + 5^2 \cdot \begin{bmatrix} -\frac{7}{65} \\ \frac{117}{680} \\ \frac{1}{1} \end{bmatrix} \right] - \frac{1}{1!} \left[0 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 2 \cdot \begin{bmatrix} \frac{3}{13} \\ 0 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} \frac{30}{13} \\ \frac{3}{34} \\ 0 \end{bmatrix} + 4 \cdot \begin{bmatrix} 0 \\ \frac{15}{17} \\ \frac{30}{691} \end{bmatrix} + 5 \cdot \begin{bmatrix} 0 \\ 0 \\ \frac{300}{691} \end{bmatrix} \right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$C_3 = \sum_{j=0}^5 \frac{j^3}{3!} A_j - \frac{1}{2!} \sum_{j=0}^5 j^2 B_j,$$

$$= \frac{1}{3!} ((0)^3 A_0 + (1)^3 A_1 + (2)^3 A_2 + (3)^3 A_3 + (4)^3 A_4 + (5)^3 A_5) - \frac{1}{2!} ((0)^2 B_0 + (1)^2 B_1 + (2)^2 B_2 + (3)^2 B_3 + (4)^2 B_4 + (5)^2 B_5),$$

$$= \frac{1}{3!} \left[0^3 \cdot \begin{bmatrix} -\frac{17}{260} \\ 7 \\ \frac{170}{117} \\ -\frac{1382}{1} \end{bmatrix} + 1^3 \cdot \begin{bmatrix} \frac{6}{13} \\ -\frac{37}{37} \\ -\frac{136}{365} \\ \frac{691}{691} \end{bmatrix} + 2^3 \cdot \begin{bmatrix} -\frac{31}{13} \\ \frac{27}{27} \\ \frac{34}{970} \\ -\frac{691}{691} \end{bmatrix} + 3^3 \cdot \begin{bmatrix} \frac{1}{59} \\ -\frac{34}{34} \\ \frac{1440}{691} \\ \frac{691}{691} \end{bmatrix} + 4^3 \cdot \begin{bmatrix} \frac{57}{52} \\ 1 \\ -\frac{2935}{1382} \\ \frac{691}{691} \end{bmatrix} + 5^3 \cdot \begin{bmatrix} -\frac{7}{65} \\ \frac{117}{680} \\ \frac{691}{691} \\ \frac{691}{691} \end{bmatrix} \right]$$

$$- \frac{1}{2!} \left[0^2 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 1^2 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 2^2 \cdot \begin{bmatrix} \frac{3}{13} \\ 0 \\ 0 \end{bmatrix} + 3^2 \cdot \begin{bmatrix} \frac{30}{13} \\ 3 \\ \frac{34}{0} \end{bmatrix} + 4^2 \cdot \begin{bmatrix} \frac{0}{15} \\ \frac{17}{17} \\ \frac{30}{30} \\ \frac{691}{691} \end{bmatrix} + 5^2 \cdot \begin{bmatrix} 0 \\ 0 \\ \frac{300}{691} \\ \frac{691}{691} \end{bmatrix} \right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$C_4 = \sum_{j=0}^5 \frac{j^4}{4!} A_j - \frac{1}{3!} \sum_{j=0}^5 J^3 B_j,$$

$$= \frac{1}{4!} ((0)^4 A_0 + (1)^4 A_1 + (2)^4 A_2 + (3)^4 A_3 + (4)^4 A_4 + (5)^4 A_5) - \frac{1}{3!} ((0)^3 B_0 + (1)^3 B_1 + (2)^3 B_2 + (3)^3 B_3 + (4)^3 B_4 + (5)^3 B_5),$$

$$= \frac{1}{4!} \left[0^4 \cdot \begin{bmatrix} -\frac{17}{260} \\ 7 \\ \frac{170}{117} \\ -\frac{1382}{1} \end{bmatrix} + 1^4 \cdot \begin{bmatrix} \frac{6}{13} \\ -\frac{37}{37} \\ -\frac{136}{365} \\ \frac{691}{691} \end{bmatrix} + 2^4 \cdot \begin{bmatrix} -\frac{31}{13} \\ \frac{27}{27} \\ \frac{34}{970} \\ -\frac{691}{691} \end{bmatrix} + 3^4 \cdot \begin{bmatrix} \frac{1}{59} \\ -\frac{34}{34} \\ \frac{1440}{691} \\ \frac{691}{691} \end{bmatrix} + 4^4 \cdot \begin{bmatrix} \frac{57}{52} \\ 1 \\ -\frac{2935}{1382} \\ \frac{691}{691} \end{bmatrix} + 5^4 \cdot \begin{bmatrix} -\frac{7}{65} \\ \frac{117}{680} \\ \frac{691}{691} \\ \frac{691}{691} \end{bmatrix} \right]$$

$$- \frac{1}{3!} \left[0^3 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 1^3 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 2^3 \cdot \begin{bmatrix} \frac{3}{13} \\ 0 \\ 0 \end{bmatrix} + 3^3 \cdot \begin{bmatrix} \frac{30}{13} \\ 3 \\ \frac{34}{0} \end{bmatrix} + 4^3 \cdot \begin{bmatrix} \frac{0}{15} \\ \frac{17}{17} \\ \frac{30}{30} \\ \frac{691}{691} \end{bmatrix} + 5^3 \cdot \begin{bmatrix} 0 \\ 0 \\ \frac{300}{691} \\ \frac{691}{691} \end{bmatrix} \right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$C_5 = \sum_{j=0}^5 \frac{j^5}{5!} A_j - \frac{1}{4!} \sum_{j=0}^5 J^4 B_j,$$

$$= \frac{1}{5!} ((0)^5 A_0 + (1)^5 A_1 + (2)^5 A_2 + (3)^5 A_3 + (4)^5 A_4 + (5)^5 A_5) - \frac{1}{4!} ((0)^4 B_0 + (1)^4 B_1 + (2)^4 B_2 + (3)^4 B_3 + (4)^4 B_4 + (5)^4 B_5),$$

$$= \frac{1}{5!} \left[0^5 \cdot \begin{bmatrix} -\frac{17}{260} \\ 7 \\ \frac{170}{117} \\ -\frac{1382}{1} \end{bmatrix} + 1^5 \cdot \begin{bmatrix} \frac{6}{13} \\ -\frac{37}{37} \\ -\frac{136}{365} \\ \frac{691}{691} \end{bmatrix} + 2^5 \cdot \begin{bmatrix} -\frac{31}{13} \\ \frac{27}{27} \\ \frac{34}{970} \\ -\frac{691}{691} \end{bmatrix} + 3^5 \cdot \begin{bmatrix} \frac{1}{59} \\ -\frac{34}{34} \\ \frac{1440}{691} \\ \frac{691}{691} \end{bmatrix} + 4^5 \cdot \begin{bmatrix} \frac{57}{52} \\ 1 \\ -\frac{2935}{1382} \\ \frac{691}{691} \end{bmatrix} + 5^5 \cdot \begin{bmatrix} -\frac{7}{65} \\ \frac{117}{680} \\ \frac{691}{691} \\ \frac{691}{691} \end{bmatrix} \right]$$

$$- \frac{1}{4!} \left[0^4 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 1^4 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 2^4 \cdot \begin{bmatrix} \frac{3}{13} \\ 0 \\ 0 \end{bmatrix} + 3^4 \cdot \begin{bmatrix} \frac{30}{13} \\ 3 \\ \frac{34}{0} \end{bmatrix} + 4^4 \cdot \begin{bmatrix} \frac{0}{15} \\ \frac{17}{17} \\ \frac{30}{30} \\ \frac{691}{691} \end{bmatrix} + 5^4 \cdot \begin{bmatrix} 0 \\ 0 \\ \frac{300}{691} \\ \frac{691}{691} \end{bmatrix} \right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$C_6 = \sum_{j=0}^5 \frac{j^6}{6!} A_j - \frac{1}{5!} \sum_{j=0}^5 J^5 B_j,$$

$$= \frac{1}{6!} ((0)^6 A_0 + (1)^6 A_1 + (2)^6 A_2 + (3)^6 A_3 + (4)^6 A_4 + (5)^6 A_5) - \frac{1}{5!} ((0)^5 B_0 + (1)^5 B_1 + (2)^5 B_2 + (3)^5 B_3 + (4)^5 B_4 + (5)^5 B_5),$$

$$\begin{aligned}
&= \frac{1}{6!} \left[0^6 \cdot \begin{bmatrix} -\frac{17}{260} \\ 7 \\ 170 \\ 117 \\ -\frac{117}{1382} \end{bmatrix} + 1^6 \cdot \begin{bmatrix} \frac{6}{13} \\ -\frac{37}{37} \\ -\frac{136}{365} \\ \frac{691}{691} \end{bmatrix} + 2^6 \cdot \begin{bmatrix} -\frac{31}{13} \\ \frac{27}{27} \\ \frac{34}{970} \\ -\frac{691}{691} \end{bmatrix} + 3^6 \cdot \begin{bmatrix} \frac{1}{59} \\ -\frac{34}{34} \\ \frac{1440}{691} \end{bmatrix} + 4^6 \cdot \begin{bmatrix} \frac{57}{52} \\ 1 \\ -\frac{2935}{1382} \end{bmatrix} + 5^6 \cdot \begin{bmatrix} -\frac{7}{65} \\ \frac{117}{680} \\ 1 \end{bmatrix} \right] \\
&\quad - \frac{1}{5!} \left[0^5 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 1^5 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 2^5 \cdot \begin{bmatrix} \frac{3}{0} \\ \frac{13}{0} \\ 0 \end{bmatrix} + 3^5 \cdot \begin{bmatrix} \frac{30}{3} \\ \frac{13}{34} \\ 0 \end{bmatrix} + 4^5 \cdot \begin{bmatrix} \frac{0}{15} \\ \frac{17}{30} \\ \frac{691}{691} \end{bmatrix} + 5^5 \cdot \begin{bmatrix} 0 \\ 0 \\ \frac{300}{691} \end{bmatrix} \right] \\
&= \begin{bmatrix} \frac{9}{-260} \\ 19 \\ \frac{680}{49} \\ -\frac{691}{691} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

Thus, the method (9) is of order five with constant error given by:

$$C_6 = \begin{bmatrix} \frac{9}{-260} \\ 19 \\ \frac{680}{49} \\ -\frac{691}{691} \end{bmatrix}$$

Definition 2 (Consistency): A linear multistep method is said to be consistent if it has order of at least one. It also follows that a LMM is consistent if and only if the following conditions are satisfied [13]:

- i. $\sum_{j=0}^k A_j = 0$,
- ii. $\sum_{j=0}^k jA_j = \sum_{j=0}^k B_j$.

Lemma 1: The implicit 3-point super class of block backward differentiation formula (I3SBPDF) scheme is consistent.

Proof:

It is shown that I3SBPDF satisfies the consistency conditions given in definition 2. From the discussion in Section 3, we deduced that the order of the method is five which is greater than one.

Let $A_0, A_1, A_2, A_3, A_4, A_5, B_0, B_1, B_2, B_3, B_4$, and B_5 be as previously defined. Then,

$$\begin{aligned}
\sum_{j=0}^5 A_j &= A_0 + A_1 + A_2 + A_3 + A_4 + A_5, \\
&= \begin{bmatrix} -\frac{17}{260} \\ 7 \\ 170 \\ 117 \\ -\frac{117}{1382} \end{bmatrix} + \begin{bmatrix} \frac{6}{13} \\ -\frac{37}{37} \\ -\frac{136}{365} \\ \frac{691}{691} \end{bmatrix} + \begin{bmatrix} -\frac{31}{13} \\ \frac{27}{27} \\ \frac{34}{970} \\ -\frac{691}{691} \end{bmatrix} + \begin{bmatrix} \frac{1}{59} \\ -\frac{34}{34} \\ \frac{1440}{691} \end{bmatrix} + \begin{bmatrix} \frac{57}{52} \\ 1 \\ -\frac{2935}{1382} \end{bmatrix} + \begin{bmatrix} -\frac{7}{65} \\ \frac{117}{680} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

Therefore the first consistency condition in (i) is satisfied.

Similarly,

$$\sum_{j=0}^5 jA_j = 0.A_0 + 1.A_1 + 2.A_2 + 3.A_3 + 4.A_4 + 5.A_5,$$

$$0. \begin{bmatrix} -\frac{17}{260} \\ \frac{7}{170} \\ \frac{117}{1382} \end{bmatrix} + 1. \begin{bmatrix} \frac{6}{13} \\ \frac{37}{136} \\ \frac{365}{691} \end{bmatrix} + 2. \begin{bmatrix} \frac{31}{13} \\ \frac{27}{34} \\ \frac{970}{691} \end{bmatrix} + 3. \begin{bmatrix} \frac{1}{59} \\ \frac{34}{1440} \\ \frac{691}{691} \end{bmatrix} + 4. \begin{bmatrix} \frac{57}{52} \\ \frac{1}{2935} \\ \frac{1382}{1382} \end{bmatrix} + 5. \begin{bmatrix} \frac{7}{65} \\ \frac{117}{680} \\ \frac{1}{691} \end{bmatrix} = \begin{bmatrix} \frac{33}{13} \\ \frac{33}{34} \\ \frac{330}{691} \end{bmatrix}.$$

$$\sum_{j=0}^5 B_j = B_0 + B_1 + B_2 + B_3 + B_4 + B_5,$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{3}{13} \\ \frac{37}{136} \\ \frac{365}{691} \end{bmatrix} + \begin{bmatrix} \frac{30}{13} \\ \frac{27}{34} \\ \frac{970}{691} \end{bmatrix} + \begin{bmatrix} \frac{0}{59} \\ \frac{1}{1440} \\ \frac{691}{691} \end{bmatrix} + \begin{bmatrix} \frac{0}{52} \\ \frac{1}{2935} \\ \frac{1382}{1382} \end{bmatrix} + \begin{bmatrix} \frac{0}{65} \\ \frac{117}{680} \\ \frac{1}{691} \end{bmatrix} = \begin{bmatrix} \frac{33}{13} \\ \frac{33}{34} \\ \frac{330}{691} \end{bmatrix}.$$

Thus, since $\sum_{j=0}^5 jA_j = \sum_{j=0}^5 B_j$, then the second consistency condition in (ii) is satisfied. Hence, we concluded that the I3SBBDF is consistent since the consistency conditions given in Definition 2 are therefore met.

4. Stability of the Method

This section presents the region of absolute stability and the zero-stability of the method by substituting the Dahlquist test differential equation of the form $y' = \lambda y$, where the parameter λ is a complex constant with negative real part into the Equation (9) to obtain:

$$\begin{aligned} y_{n+1} &= \frac{17}{260}y_{n-2} - \frac{6}{13}y_{n-1} + \frac{31}{13}y_n - \frac{57}{52}y_{n+2} + \frac{7}{65}y_{n+3} + \frac{30}{13}h\lambda y_{n+1} + \frac{3}{13}h\lambda y_n, \\ y_{n+2} &= -\frac{59}{170}y_{n-2} + \frac{37}{136}y_{n-1} - \frac{27}{34}y_n + \frac{117}{34}y_{n+1} - \frac{117}{680}y_{n+3} + \frac{15}{17}h\lambda y_{n+2} + \frac{3}{34}h\lambda y_{n+1}, \\ y_{n+3} &= \frac{1440}{691}y_{n-2} - \frac{2935}{691}y_{n-1} + \frac{970}{691}y_n - \frac{1440}{691}y_{n+1} + \frac{2935}{1382}y_{n+2} + \frac{300}{691}h\lambda y_{n+3} + \frac{30}{691}h\lambda y_{n+2}. \end{aligned} \tag{15}$$

By rearranging and collecting the like terms we get:

$$\begin{aligned} \left(1 - \frac{30}{13}h\lambda\right)y_{n+1} + \frac{57}{52}y_{n+2} - \frac{7}{65}y_{n+3} &= \frac{17}{260}y_{n-2} - \frac{6}{13}y_{n-1} + \left(\frac{31}{13} + \frac{3}{13}h\lambda\right)y_n, \\ \left(-\frac{59}{34} - \frac{3}{34}h\lambda\right)y_{n+1} + \left(1 - \frac{15}{17}h\lambda\right)y_{n+2} + \frac{117}{680}y_{n+3} &= -\frac{7}{170}y_{n-2} + \frac{37}{136}y_{n-1} - \frac{27}{34}y_n, \\ \frac{1440}{691}y_{n+1} + \left(-\frac{2935}{1382} - \frac{30}{691}h\lambda\right)y_{n+2} + \left(1 - \frac{300}{691}h\lambda\right)y_{n+3} &= \frac{117}{1382}y_{n-2} - \frac{365}{691}y_{n-1} + \frac{970}{691}y_n. \end{aligned} \tag{16}$$

The matrix formulation of (16) leads to:

$$\begin{bmatrix} \left(1 - \frac{30}{13}h\lambda\right) & \frac{57}{52} & -\frac{7}{65} \\ \left(-\frac{59}{34} - \frac{3}{34}h\lambda\right) & \left(1 - \frac{15}{17}h\lambda\right) & \frac{117}{680} \\ \frac{1440}{691} & \left(-\frac{2935}{1382} - \frac{30}{691}h\lambda\right) & \left(1 - \frac{300}{691}h\lambda\right) \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix} = \begin{bmatrix} \frac{17}{260} & -\frac{6}{13} & \left(\frac{31}{13} + \frac{3}{13}h\lambda\right) \\ -\frac{7}{170} & \frac{37}{136} & -\frac{27}{34} \\ \frac{117}{1382} & -\frac{365}{691} & \frac{970}{691} \end{bmatrix} \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix}. \tag{17}$$

Putting $\bar{h} = h\lambda$ in (17), we have

$$\begin{bmatrix} \left(1 - \frac{30}{13}\bar{h}\right) & \frac{57}{52} & -\frac{7}{65} \\ \left(-\frac{59}{34} - \frac{3}{34}\bar{h}\right) & \left(1 - \frac{15}{17}\bar{h}\right) & \frac{117}{680} \\ \frac{1440}{691} & \left(-\frac{2935}{1382} - \frac{30}{691}\bar{h}\right) & \left(1 - \frac{300}{691}\bar{h}\right) \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix} = \begin{bmatrix} \frac{17}{260} & -\frac{6}{13} & \left(\frac{31}{13} + \frac{3}{13}\bar{h}\right) \\ -\frac{7}{170} & \frac{37}{136} & -\frac{27}{34} \\ \frac{117}{1382} & -\frac{365}{691} & \frac{970}{691} \end{bmatrix} \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix}. \tag{18}$$

Definition 3: If m is the number of block and r is the number of points in the block, then $n = mr$. Here $r = 3$ and $n = 3m$. By [14], we let

$$Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix} = \begin{bmatrix} y_{3m+1} \\ y_{3m+2} \\ y_{3m+3} \end{bmatrix}, Y_{m-1} = \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} y_{3m-2} \\ y_{3m-1} \\ y_{3m} \end{bmatrix} = \begin{bmatrix} y_{3(m-1)+1} \\ y_{3(m-1)+2} \\ y_{3(m-1)+3} \end{bmatrix}.$$

The matrix equation (18) can also be written as:

$$AY_m = BY_{m-1}, \tag{19}$$

Where

$$A = \begin{bmatrix} \left(1 - \frac{30}{13}\bar{h}\right) & \frac{57}{52} & -\frac{7}{65} \\ \left(-\frac{59}{34} - \frac{3}{34}\bar{h}\right) & \left(1 - \frac{15}{17}\bar{h}\right) & \frac{117}{680} \\ \frac{1440}{691} & \left(-\frac{2935}{1382} - \frac{30}{691}\bar{h}\right) & \left(1 - \frac{300}{691}\bar{h}\right) \end{bmatrix}, B = \begin{bmatrix} \frac{17}{260} & -\frac{6}{13} & \left(\frac{31}{13} + \frac{3}{13}\bar{h}\right) \\ \frac{7}{170} & \frac{37}{136} & \frac{27}{34} \\ \frac{117}{1382} & -\frac{365}{691} & \frac{970}{691} \end{bmatrix}.$$

The stability polynomial of the method which the combination of both first and second characteristics polynomial is found by evaluating

$$\pi(t, \bar{h}) = \det(At + B) = 0. \tag{20}$$

To obtain the following stability polynomial as:

$$\pi(t, \bar{h}) = \frac{392985}{1221688}\bar{h}t + \frac{1006023}{2443376}t + \frac{723}{2443376}\bar{h} + \frac{355}{305422} + \frac{8522833}{2443376}t^3 - \frac{2289015}{610844}\bar{h}^2t^2 - \frac{10088235}{2443376}\bar{h}t^2 - \frac{35043}{8983}t^2 - \frac{135000}{152711}\bar{h}^3t^3 + \frac{513342}{152711}\bar{h}^2t^3 - \frac{97293}{17966}\bar{h}t^3 + \frac{4338}{152711}\bar{h}^2t - \frac{135}{152711}\bar{h}^3t^2 = 0, \tag{21}$$

The region of absolute stability (RAS) of the method is drawn in Maple software environment by substituting $t = e^{i\theta}$ for $0 \leq \theta \leq 2\pi$ into the stability polynomial (21) as shown in Fig. 1.

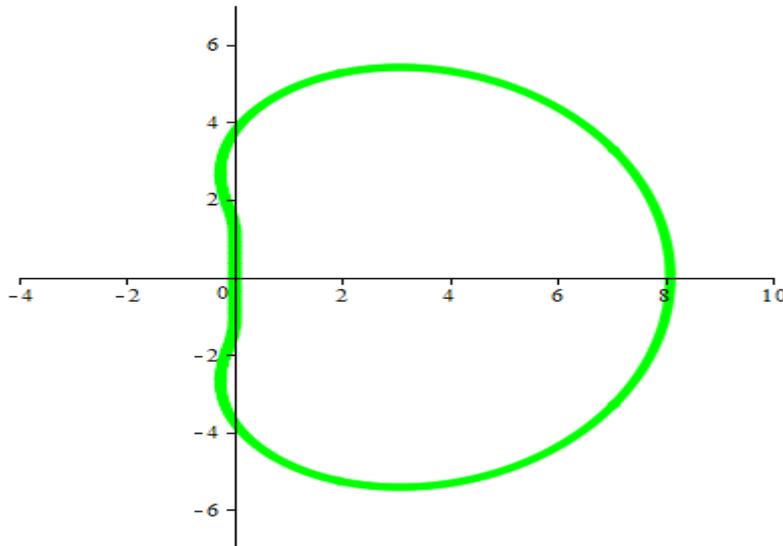


Fig. 1 Stability region for the 3-point when $\rho = \frac{1}{10}$.

Definition 4 (A-stability): A linear multistep method (LMM) is said to be A-stable if its stability region covers the entire left- hand half of the complex plane [15].

Definition 5 (A(α)-stability): A linear multistep method is said to be almost A(α)-stable, $\alpha \in (0, \frac{\pi}{2})$ if all solutions of $y' = \lambda y$ converge to 0 as n tends to infinity with a fixed h , so that $|\arg(-\lambda)| < \alpha, |\lambda| \neq 0$ [16].

Hence, in accordance with Definition 4, the method (9) is almost A-stable since its stability region enclosed almost the whole left-hand half of the complex plane, thus having satisfied this criterion for A-stability, the method is suitable for the integration of stiff problems.

To demonstrate the zero stability of the method, we set $\bar{h} = 0$ in (21), resulting in the following first characteristic polynomial:

$$\frac{355}{305422} + \frac{1006023}{2443376}t + \frac{8522833}{2443376}t^3 - \frac{35043}{8983}t^2 = 0. \tag{22}$$

By solving equation (22) for t , we get:

$$t = 1, t = -0.00275111089, t = 0.1211229011.$$

Definition 6 (Zero-stability): The Equation (13) is said to be zero stable if no root of the first characteristic polynomial has modulus greater than one and that any root with modulus one is simple [17]. Therefore, according to the above definition, the method (9) is zero stable.

5. Implementation of the Method

Newton’s iteration is used to implement the method as described below. We start by defining the error.

Definition 7: Let y_i and $y(x_i)$ be the approximate and exact solutions, respectively. The absolute error is defined by

$$(\text{error}_i)_t = |(y_i)_t - (y(x_i))_t| \tag{23}$$

The maximum error is given by

$$\text{MAXE} = \max_{1 \leq i \leq T} \left(\max_{1 \leq i \leq N} (\text{error}_i)_t \right) \tag{24}$$

where T is the total number of steps and N is the number of equations.

Let $y_{n+1}^{(i+1)}$ denote the $(i + 1)^{th}$ iterate and

$$y_{n+j}^{(i+1)} = y_{n+j}^{(i+j)} - y_{n+j}^{(i)}, \quad j = 1,2,3. \tag{25}$$

Let

$$\begin{aligned} F_1 &= y_{n+2} + \frac{57}{52}y_{n+2} - \frac{7}{65}y_{n+3} - \frac{30}{13}hf_{n+1} - \frac{3}{13}hf_n - \xi_1, \\ F_2 &= y_{n+2} - \frac{59}{34}y_{n+1} + \frac{117}{680}y_{n+3} - \frac{15}{17}hf_{n+2} - \frac{3}{34}hf_{n+1} - \xi_2, \\ F_3 &= y_{n+3} + \frac{1440}{691}y_{n+1} - \frac{2935}{1382}y_{n+2} - \frac{300}{691}hf_{n+3} - \frac{30}{691}hf_{n+2} - \xi_3, \end{aligned} \tag{26}$$

Where

$$\begin{aligned} \xi_1 &= \left(\frac{17}{260}y_{n-2} - \frac{6}{13}y_{n-1} + \frac{31}{13}y_n \right), \\ \xi_2 &= \left(-\frac{7}{170}y_{n-2} + \frac{37}{136}y_{n-1} - \frac{27}{34}y_n \right), \\ \xi_3 &= \left(\frac{117}{1382}y_{n-2} - \frac{365}{691}y_{n-1} + \frac{970}{691}y_n \right). \end{aligned} \tag{27}$$

Then, the iteration takes the form:

$$y_{n+j}^{(i+1)} = y_{n+j}^{(i)} - [F_j(y_{n+j}^{(i)})][F_j'(y_{n+j}^{(i)})]^{-1}, \quad j = 1,2,3, \tag{28}$$

which can be expressed as:

$$[F_j'(y_{n+j}^{(i)})]e_{n+j}^{(i+1)} = -[F_j(y_{n+j}^{(i)})], \quad j = 1,2,3. \tag{29}$$

Equation (29) can be expressed in the following matrix form:

$$\begin{bmatrix} 1 - \frac{30}{13}h \frac{\delta f_{n+1}}{\delta y_{n+1}} & \frac{57}{52} & -\frac{7}{65} \\ -\frac{59}{34} - \frac{3}{34}h \frac{\delta f_{n+1}}{\delta y_{n+1}} & 1 - \frac{15}{17}h \frac{\delta f_{n+2}}{\delta y_{n+2}} & \frac{117}{680} \\ \frac{1440}{691} & -\frac{2935}{1382} - \frac{30}{691}h \frac{\delta f_{n+2}}{\delta y_{n+2}} & 1 - \frac{300}{691}h \frac{\delta f_{n+3}}{\delta y_{n+3}} \end{bmatrix} \begin{bmatrix} e_{n+1}^{(i+1)} \\ e_{n+2}^{(i+1)} \\ e_{n+3}^{(i+1)} \end{bmatrix} = \begin{bmatrix} 1 & \frac{57}{52} & -\frac{7}{65} \\ -\frac{59}{34} & 1 & \frac{117}{680} \\ \frac{1440}{691} & -\frac{2935}{1382} & 1 \end{bmatrix} \begin{bmatrix} y_{n+1}^{(i)} \\ y_{n+2}^{(i)} \\ y_{n+3}^{(i)} \end{bmatrix} + h \begin{bmatrix} 0 & 0 & \frac{3}{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n-2}^{(i)} \\ f_{n-1}^{(i)} \\ f_n^{(i)} \end{bmatrix} + h \begin{bmatrix} \frac{30}{13} & 0 & 0 \\ \frac{3}{34} & \frac{15}{17} & 0 \\ 0 & \frac{30}{691} & \frac{300}{691} \end{bmatrix} \begin{bmatrix} f_{n+1}^{(i)} \\ f_{n+2}^{(i)} \\ f_{n+3}^{(i)} \end{bmatrix} + \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}. \quad (30)$$

A computer code in C-programming language is developed for the implementation of equation (25).

6. Test Problems and Numerical Results

To assess the effectiveness of the proposed method, the following oscillatory and stiff ordinary differential equations are solved:

Problem 1:

$$y' = -20y + 20 + \sin x + \cos x, \quad y(0) = 1, \quad 0 \leq x \leq 2.$$

Exact Solution:

$$y(x) = \sin x + e^{-20x}.$$

Source: [18]

Problem 2:

$$\begin{aligned} y_1' &= 9y_1 + 24y_2 + 5\cos x - \frac{1}{3}\sin x, & y_1(0) &= \frac{4}{3}, & 0 \leq x \leq 10. \\ y_2' &= -24y_1 - 51y_2 - 9\cos x + \frac{1}{3}\sin x, & y_2(0) &= \frac{2}{3}, \end{aligned}$$

Exact Solution:

$$\begin{aligned} y_1(x) &= 2e^{-3x} - e^{-39x} + \frac{1}{3}\cos x, \\ y_2(x) &= -e^{-3x} + 2e^{-39x} - \frac{1}{3}\cos x. \end{aligned}$$

Eigen values:

$$\lambda = -3 \text{ and } \lambda = -39.$$

Source: [19]

Problem 3:

$$\begin{aligned} y_1' &= 32y_1 + 66y_2 + \frac{2}{3}x + \frac{2}{3}, & y_1(0) &= \frac{1}{3}, & 0 \leq x \leq 1. \\ y_2' &= -66y_1 - 133y_2 - \frac{1}{3}x - \frac{1}{3}, & y_2(0) &= \frac{1}{3}, \end{aligned}$$

Exact Solution:

$$\begin{aligned} y_1(x) &= \frac{2}{3}x + \frac{2}{3}e^{-x} - \frac{1}{3}e^{-100x}, \\ y_2(x) &= -\frac{1}{3}x - \frac{1}{3}e^{-x} + \frac{2}{3}e^{-100x}. \end{aligned}$$

Eigen values:

$$\lambda = -1 \text{ and } \lambda = -100,$$

Source: [20]

The problems outlined above have been addressed through the application of the developed method, employing both the 3-point BBDF and the 3-point SBBDF with various step sizes (h). The tables below present details on the maximum error and computation time for each problem, accompanied by the use of specified notations.

h =step size;

3BBDF = 3-point BBDF method [5];

3SBBDF=3-point SBBDF method [7];

I3SBBDF=Improved 3-point SBBDF method;

NS=Total number of integration steps;

MAXE=Maximum Error;

Time=Computation time.

Table 2 Numerical results for problem 1

H	METHOD	NS	MAXE	TIME
10^{-2}	3BBDF	66	7.16194e+008	2.66800e-002
	3SBBDF	66	3.70158e-002	2.56500e-002
	I3SBBDF	66	3.24894e-002	2.67100e-002
10^{-3}	3BBDF	666	2.07885e-002	2.51600e-002
	3SBBDF	666	7.33017e-004	2.56700e-002
	I3SBBDF	666	6.00560e-004	2.58400e-002
10^{-4}	3BBDF	6666	2.19481e-003	2.77100e-002
	3SBBDF	6666	8.24480e-006	3.36900e-002
	I3SBBDF	6666	6.38650e-006	3.43800e-002
10^{-5}	3BBDF	66666	2.20579e-004	6.27900e-002
	3SBBDF	66666	8.38946e-008	1.10500e-001
	I3SBBDF	66666	6.46216e-008	1.10000e-001
10^{-6}	3BBDF	666666	2.20688e-005	4.01700e-001
	3SBBDF	666666	8.40897e-010	8.72800e-001
	I3SBBDF	666666	6.47340e-010	8.82300e-001

Table 3 Numerical results for problem 2

H	METHOD	NS	MAXE	TIME
10^{-2}	3BBDF	333	6.62694e+099	2.44900e-002
	3SBBDF	333	1.37962e-001	3.61700e-002
	I3SBBDF	333	1.20445e-001	3.95500e-002
10^{-3}	3BBDF	3333	7.44768e-002	3.50800e-002
	3SBBDF	3333	5.04589e-003	6.34700e-002
	I3SBBDF	3333	4.27492e-003	5.60800e-002
10^{-4}	3BBDF	33333	8.45376e-003	9.53700e-002
	3SBBDF	33333	6.15656e-005	2.69400e-001
	I3SBBDF	33333	4.80211e-005	2.60400e-001
10^{-5}	3BBDF	333333	8.53717e-004	6.44100e-001
	3SBBDF	333333	6.34826e-007	2.53200e+000
	I3SBBDF	333333	4.89186e-007	2.46400e+000
10^{-6}	3BBDF	3333333	8.54545e-005	6.06100e+000
	3SBBDF	3333333	6.37502e-009	2.21800e+001
	I3SBBDF	3333333	4.90786e-009	2.22000e+001

Table 4 Numerical results for problem 3

H	METHOD	NS	MAXE	TIME
10^{-2}	3BBDF	33	1.95803e+002	2.38700e-002
	3SBBDf	33	1.21523e-002	2.71900e-002
	I3SBBDf	33	1.21587e-002	3.87700e-002
10^{-3}	3BBDF	333	4.61401e-002	2.33000e-002
	3SBBDf	333	8.66386e-003	2.62200e-002
	I3SBBDf	333	7.65746e-003	3.44100e-002
10^{-4}	3BBDF	3333	7.14998e-003	2.62500e-002
	3SBBDf	3333	1.29624e-004	3.18700e-002
	I3SBBDf	3333	1.03443e-004	4.03000e-002
10^{-5}	3BBDF	33333	7.33633e-004	5.28000e-002
	3SBBDf	33333	1.38655e-006	9.96200e-002
	I3SBBDf	33333	1.07100e-006	1.09000e-001
10^{-6}	3BBDF	333333	7.35458e-005	2.94700e-001
	3SBBDf	333333	1.39989e-008	7.55300e-001
	I3SBBDf	333333	1.07797e-008	8.02400e-001

A graphical representation in Fig. 2-4 illustrates the accuracy of these methods, plotting $\text{Log}_{10}(\text{MAXE})$ against H for each tested problem.

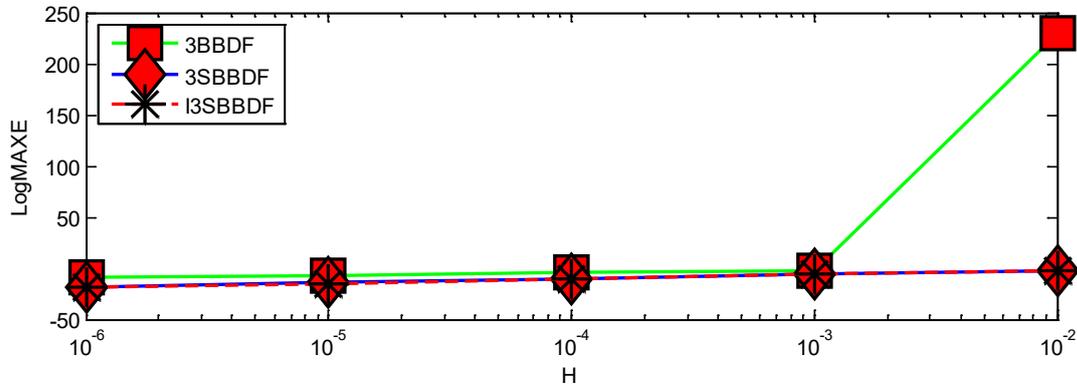


Fig. 2 Graph of $\text{LOG}_{10}(\text{MAXE})$ against H for problem 1

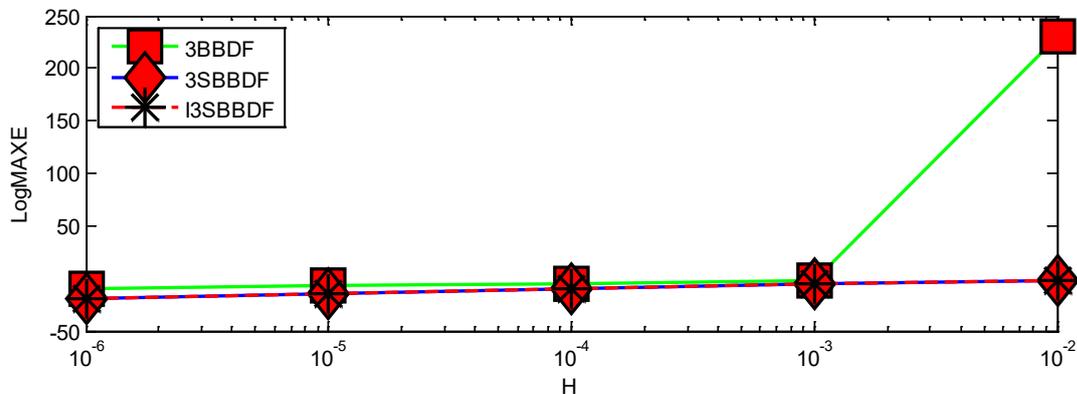


Fig. 3 Graph of $\text{LOG}_{10}(\text{MAXE})$ against H for problem 2

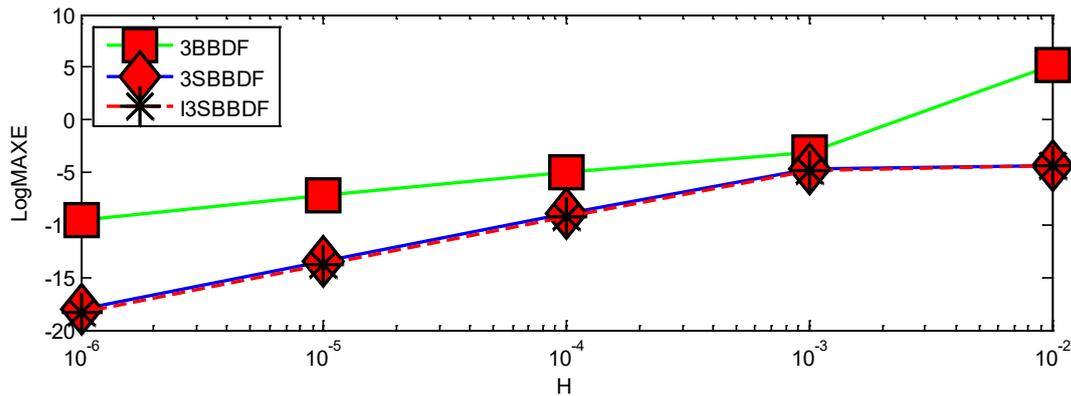


Fig. 4 Graph of $LOG_{10}(MAXE)$ against H for problem 3

7. Discussion on the Result

The numerical results presented in the above tables showed that the I3SBBDf method achieves smaller maximum error across all the step sizes compared to the conventional 3BBDF and 3SBBDf methods, confirming its reliability for integrating stiff and oscillatory ODEs.

The graphs further indicate that the I3SBBDf method maintains lower scaled errors and smoother error curves across all tested problems, reflecting its enhanced accuracy and stability. This improvement is mainly due to the implicit block structure, which effectively controls error propagation and handles stiffness more efficiently than the existing BBDF variants.

Overall, the I3SBBDf method demonstrates a strong balance between stability, accuracy, and computational efficiency, making it a reliable alternative for stiff initial value problems.

Conclusion

A novel fifth-order method has been developed for integrating stiff ordinary differential equations. This approach incorporates the block backward differentiation formula as a special case and generates three solution values per step. By adjusting the parameter within the range (-1,1), different variations of the formula can be obtained. When set to a specific value of $\rho = \frac{1}{10}$, the method exhibits almost A-stability and demonstrates higher accuracy than the 3SBBDf and 3BBDF methods. Additionally, it is zero-stable and has a lower error constant compared to the 3SBBDf method. Therefore, the proposed I3SBBDf method is a reliable and efficient choice for solving stiff and oscillatory ordinary differential equations.

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Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Author Contribution

The authors confirm contribution to the paper as follows: **Study conception and design:** Adamu Abdulrahman, Alhassan Buhari; **data collection:** Adamu Abdulrahman; **analysis and interpretation of results:** Adamu Abdulrahman, Alhassan Buhari, Musa Hamisu, Aliyu Ja’afar; **draft manuscript preparation:** Adamu Abdulrahman, Alhassan Buhari. All authors reviewed the results and approved the final version of the manuscript.

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