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# On the Numerical Solution of the Nonlinear Stochastic Kadar-Parisi-Zhang Equation with Reduced Differential Transform Method

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**Abstract**

The Reduced Differential Transform Method (RDTM) was used to obtain approximate solutions to the Kadar-Parisi-Zhang equation. The Kadar-Parisi-Zhang equation which is a stochastic nonlinear partial differential equation was used to investigate the growth (or erosion) of interfaces by particle deposition for bounded noise for an initially sinusoidal surface. The solutions obtained through RDTM with Mathematica package was compared to the finite difference solutions. The study shows the efficacy of the RDTM in solving the Kadar-Parisi-Zhang equation and is therefore a wonderful tool for solving nonlinear partial differential equations of stochastic or deterministic nature.

**Keywords:** RDTM; kadar-parisi-zhang equation; surface growth; particle deposition

## 1. INTRODUCTION

A stochastic partial differential equation (SPDE) is a partial differential equation containing a random (noise) term. Stochastic partial differential equations appear in several different applications including the study of random evolution of systems (random interface growth, random evolution of surfaces, fluids subject to random forcing). They include the Edwards-Wilkinson (EW), the Kadar-Parisi Zhang (KPZ) and the Cuerno-Barabasi equations.

The goal of differential equations in surface growth is to find the variation with time of the interface height $h(x,t)$ at any position x. The Edwards-Wilkinson equation which was the first continuum equation used to study the growth of interfaces by particle deposition is given by:

$$\frac{\partial h(x,t)}{\partial t} = v\nabla^2 h(x,t) + \mu(x,t) \qquad (1)$$

Here v is sometimes called a 'surface tension' for the Laplacian term $v\nabla^2 h$ tends to smooth the interface. The Laplacian term smoothens by redistributing the irregularities on the interface while maintaining the average height unchanged. Thus, the surface tension acts as a conservative relaxation mechanism [1]. The PDE is stochastic because of the presence of $\eta(x,t)$ a Gaussian and white noise with the following properties:

$$\langle \eta(x,t) \rangle = 0 \qquad (2)$$

$$\langle \eta(x,t)\eta(x^{'},t^{'}) \rangle = \Gamma^2 \delta(x-x^{'})\delta(t-t^{'}) \qquad (3)$$

This shows that the noise has zero configurational average and has no correlations in space and time. Often used in numerical simulations is 'bounded noise' in which $\eta = \pm 1$ [2].

The predictions of this linear theory change however when non-linear terms are added to the growth equation. The first extension of the EW equation to include non-linear terms was proposed by Kadar, Parisi and Zhang [3]. Lateral growth which occurs locally normal to the interface suggests that a non-linear term of form $(\nabla h)^2$ must be present in the growth equation to reflect the presence of lateral growth. Adding this term to the EW equation, we obtain the KPZ equation.

$$\frac{\partial (x,t)}{\partial t} = v\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(x,t) \qquad (4)$$

The first term on the right hand side describes relaxation of the interfaces by a surface tension v [4] (R. H. Landau and M. J. Paez, 1997).

The Reduced differential transform method (RDTM) has been applied to various nonlinear partial differential equations [5,6,7,8].

The motivation of this paper is to apply the reduced differential transform method to solve the stochastic KPZ equation for 'bounded noise' in which $\eta = \pm 1$ and compare this solutions to the corresponding finite difference solutions. The initial and boundary conditions including other necessary parameters are given below:

Initial condition $h(x,t=0) = \sin\left(\frac{\pi x}{l}\right)$

Boundary condition $h(0,t) = h(l,t) = \eta t + \frac{\lambda t}{2}\left[\frac{\pi}{l}\cos\left(\frac{\pi x}{l}\right)\right]^2$

With $\Delta x = 0.1, \Delta t = 0.1$ for $0 \le x \le 5$

The KPZ equation is solved for different values of $v$ and $\lambda$.

## 2. FINITE DIFFERENCE APPROACH

From the KPZ equation (4), we transform each term of the equation into finite difference form to obtain a general finite difference scheme for the KPZ equation.

$$\frac{\partial h(x,t)}{\partial t} = v\frac{\partial^2 h(x,t)}{\partial x^2} + \frac{\lambda}{2}\left(\frac{\partial h(x,t)}{\partial x}\right)^2 + \eta(x,t)$$

$$\frac{\partial h(x,t)}{\partial t} = \frac{h_{i,j+1} - h_{i,j}}{\Delta t}$$

$$\frac{\partial^2 h(x,t)}{\partial x^2} = \frac{h_{i,j+1} - 2h_{i,j} + h_{i-1,j}}{(\Delta x)^2}$$

$$\frac{\partial h(x,t)}{\partial x} = \frac{h_{i+1,j} - h_{i,j}}{\Delta x}$$

$$\frac{h_{i,j+1} - h_{i,j}}{\Delta t} = v\left(\frac{h_{i+1,j} - 2h_{i,j} + h_{i-1,j}}{(\Delta x)^2}\right) + \frac{\lambda}{2}\left(\frac{h_{i+1,j} - 2h_{i,j}}{\Delta x}\right)^2 + \eta$$

$$h_{i,j+1} - h_{i,j}$$
$$= \frac{v\Delta t}{(\Delta x)^2}(h_{i+1,j} - 2h_{i,j} + h_{i-1,j}) + \frac{\lambda \Delta t}{2(\Delta x)^2}(h_{i,j+1} - h_{i,j})^2 + \mu \Delta t$$
$$h_{i,j+1} = h_{i,j} + \frac{v\Delta t}{(\Delta x)^2}(h_{i+1,j} - 2h_{i,j} + h_{i-1,j}) + \frac{\lambda \Delta t}{2(\Delta x)^2}(h_{i+1,j} - h_{i,j})^2$$
$$h_{i,j+1} = h_{i,j} + A(h_{i+1,j} - 2h_{i,j} + h_{i-1,j}) + B(h_{i+1,j} - h_{i,j})^2 + C \tag{5}$$

where $A = \frac{v\Delta t}{(\Delta x)^2}, B = 2\frac{\lambda \Delta t}{(\Delta x)^2}$ and $C = \eta \Delta t$.

The finite difference formula in (5) is for the solving the one-dimensional Kadar-Parisi-Zhang equation and is implemented with FORTRAN

## 2.1 Reduced Differential Transform Method

The basic definitions of the Reduced Differential Transform Method (RDTM) are introduced as follows (Y. Keskin and G. Oturanc , 2009);

If the function $u(x,t)$ is analytic and differentiable continuously with respect to time $t$ and space $x$ in the domain of interest, then we introduce $U_k(x)$ such that,

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{t=0} \tag{6}$$

Where the $t$-dimensional spectrum function $U_k(x)$ is the transformed function of $U_k(x,t)$. The inverse differential transform of $U_k(x)$ is defined as follows;

$$u_k(x,t) = \sum_{k=0}^{\infty} U_k(x) t^k \tag{7}$$

Combining equation (6) and (7) we can write

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial u(x,t)}{\partial t^k}\right]_{t=0} t^k \tag{8}$$

From the above definitions, it can be seen that the concept of the RDTM is derived from the power series expansion. According to the Reduced Differential Transform Method and Table 2.1 from (Y. Keskin and G. Oturanc , 2009), (Y. Keskin and G. Oturanc, 2010), (Y. Keskin, S. Servi and G. Oturanc, 2011) we can construct an iteration formula to obtain values of $U_1, U_2, U_3, U_4, U_5$, etc.

The first term of the transformed solution $U_0(x)$ is just the initial condition at the time $t=0$. Also, for second order partial differential equations, the derivative of the initial condition with respect to time gives $U_1(x)$.The inverse transformation of the set of values $\{U_k(x)\}_{k=0}^{n}$ gives approximate solution as

$$u_n(x,t) = \sum_{k=0}^{n} U_k(x)t^k \tag{9}$$

Where $n$ is the order of the approximation. Therefore, the exact solution of the equation is given by;

$$u(x,t) = \lim_{n\to\infty} \overline{u_n}(x,t) \tag{10}$$

**Table 1:** Differential Transform Table

| Functional Form | Transformed Form |
|---|---|
| $u(x,t)$ | $u_k(x) = \frac{1}{k!}\left[\frac{\partial^2 u(x,t)}{\partial t^k}\right]_{t=0}$ |
| $u(x,t)\pm v(x,t)$ | $u_k(x)\pm v_k(x)$ |
| $\alpha u(x,t)$ | $\alpha U_k(x)$ (α is a constant) |
| $x^m t^n$ | $x^m\delta(k-n)$ |
| $x^m t^n u(x,t)$ | $x^m U(k-n)$ |
| $u(x,t)v(x,t)$ | $\sum_{r=0}^{k} U_r(x)V_{k-r}(x)$ |
| $\frac{\partial^r}{\partial t^r}u(x,t)$ | $\frac{(k+r)!}{k!}U_{k+r}(x)$ |
| $\frac{\partial}{\partial x}u(x,t)$ | $\frac{\partial}{\partial x}U_k(x)$ |
| Nonlinear function $N(u(x,t)) = F(u(x,t))$ | $N_k = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}F(U_{0)}\right]_{t=0}$ |

## 3. APPLICATION

Consider the Kadar-Parisi-Zhang equation (4):

$$\frac{\partial h(x,t)}{\partial t} = \nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(x,t)$$

Equation (4) can be written in standard operator form as equation (11), where $L=\frac{\partial}{\partial t}, R=v\frac{\partial^2}{\partial x^2}$ and $N=\frac{\lambda}{2}\left(\frac{\partial}{\partial x}\right)^2$ is the non-linear term.

$$L(h(x,t))=R(h(x,t))+N(h(x,t))+\eta \tag{11}$$

With the initial condition

$$h(x,0)=f(x)=\sin\left(\frac{\pi x}{l}\right)$$

From Table 1, we can transform each term of the KPZ equation as follows:

$$v\frac{\partial h(x,t)}{\partial x^2}=v\frac{\partial h_k(x)}{\partial x^2} \tag{12}$$

$$\frac{\partial h(x,t)}{\partial t}=\frac{(k+1)!}{k!}H_{k+1}(x) \tag{13}$$

$$\eta(x,t)=\eta\partial(k) \text{ where } \partial(k)\begin{cases}1 & k=0\\ 0 & otherwise\end{cases} \tag{14}$$

The following steps give first few non-linear terms $N_k$

$$N_0=\frac{1}{0!}\left[\frac{\partial^0}{\partial t^0}\left[\left(\frac{\partial H_0}{\partial x}\right)^2\right]\right]=\left(\frac{\partial H_0}{\partial x}\right)^2$$

$$N_1=\frac{1}{1!}\left[\frac{\partial}{\partial t}\left[\left(\frac{\partial H_0}{\partial x}\right)^2\right]\right]=2\frac{\partial H_0}{\partial x}\frac{\partial H_1}{\partial x}$$

$$N_2=\frac{1}{2!}\left[\frac{\partial^2}{\partial t^2}\left[\left(\frac{\partial H_0}{\partial x}\right)^2\right]\right]=\frac{1}{2!}\left[\frac{\partial}{\partial t}\left(2\frac{\partial H_0}{\partial x}\frac{\partial H_1}{\partial x}\right)\right]$$

$$N_2=2\frac{\partial H_0}{\partial x}\frac{\partial H_2}{\partial x}+\left(\frac{\partial H_1}{\partial x}\right)^2$$

$$N_3=\frac{1}{3!}\left[\frac{\partial^3}{\partial t^3}\left[\left(\frac{\partial H_0}{\partial x}\right)^2\right]\right]=\frac{1}{3!}\left[\frac{\partial}{\partial t}\left(4\frac{\partial H_0}{\partial x}\frac{\partial H_2}{\partial x}+2\left(\frac{\partial H_1}{\partial x}\right)^2\right)\right]$$

$$N_3=2\frac{\partial H_0}{\partial x}\frac{\partial H_3}{\partial x}+2\frac{\partial H_1}{\partial x}\frac{\partial H_2}{\partial x}$$

$$N_4 = \frac{1}{4!}\left[\frac{\partial^4}{\partial t^4}\left[\left(\frac{\partial H_0}{\partial x}\right)^2\right]\right] = \frac{1}{4!}\left[\frac{\partial}{\partial t}\left(12\frac{\partial H_0}{\partial x}\frac{\partial H_3}{\partial x} + 12\frac{\partial H_1}{\partial x}\frac{\partial H_2}{\partial x}\right)\right]$$

$$N_4 = 2\frac{\partial H_0}{\partial x}\frac{\partial H_4}{\partial x} + 2\frac{\partial H_1}{\partial x}\frac{\partial H_3}{\partial x} + \left(\frac{\partial H_2}{\partial x}\right)^2$$

$$N_5 = \frac{1}{5!}\left[\frac{\partial^5}{\partial t^5}\left[\left(\frac{\partial H_0}{\partial x}\right)^2\right]\right]$$

$$= \frac{1}{5!}\left[\frac{\partial}{\partial t}\left(48\frac{\partial H_0}{\partial x}\frac{\partial H_4}{\partial x} + 48\frac{\partial H_1}{\partial x}\frac{\partial H_3}{\partial x} + 24\left(\frac{\partial H_2}{\partial x}\right)^2\right)\right]$$

$$N_5 = 2\frac{\partial H_0}{\partial x}\frac{\partial H_5}{\partial x} + 2\frac{\partial H_1}{\partial x}\frac{\partial H_4}{\partial x} + 2\frac{\partial H_2}{\partial x}\frac{\partial H_3}{\partial x}$$

Substituting (12) – (14) into (11) we get the transformed form of the Kadar-Parisi-Zhang equation.

$$\frac{(k+1)!}{k!}H_{k+1}(x) = v\frac{\partial H_k(x)}{\partial x^2} + \frac{\lambda}{2}N_k + \eta\partial(k) \tag{15}$$

From the initial conditions we can write

$$H_0(x) = f(x) = \sin\left(\frac{\pi x}{l}\right)$$

Subsequent $H_k(x)$ values are obtained by the following set of equations.

$$\frac{1!}{0!}H_1(x) = v\frac{\partial H_0(x)}{\partial x^2} + \frac{\lambda}{2}N_0 + \eta\partial \tag{0}$$

$$\frac{2!}{1!}H_2(x) = v\frac{\partial H_1(x)}{\partial x^2} + \frac{\lambda}{2}N_1 + \eta\partial \tag{1}$$

$$\frac{3!}{2!}H_3(x) = v\frac{\partial H_2(x)}{\partial x^2} + \frac{\lambda}{2}N_2 + \eta\partial \tag{2}$$

$$\frac{4!}{3!}H_4(x) = v\frac{\partial H_3(x)}{\partial x^2} + \frac{\lambda}{2}N_3 + \eta\partial \tag{3}$$

$$\frac{5!}{4!}H_5(x) = v\frac{\partial H_4(x)}{\partial x^2} + \frac{\lambda}{2}N_4 + \eta\partial \tag{4}$$

$$\frac{6!}{5!}H_6(x) = v\frac{\partial H_5(x)}{\partial x^2} + \frac{\lambda}{2}N_5 + \eta\partial \tag{5}$$

The solution is given by the inverse differential transform of the $H_k(x)$ values as follows:

$$h_n(x,t) = \sum_{k=0}^{n} H_k(x)t^k = H_0(x) + H_1(x)t^1 + H_2(x)t^2 + H_3(x)t^3 + .... \qquad (16)$$

The implementation of the RDTM for the KPZ equation is done with the aid of MATHEMATICA to obtain the $U_k$ values and plug them into (16) to get the approximate solution.

## 4. RESULTS AND DISCUSSION

The reduced differential transform method has been applied to the Kadar-Parisi-Zhang equation to investigate the growth (or erosion) of interfaces by particle deposition for bounded noise in which $\eta$ takes the values +1 and -1 with the non-linear term of the form $\frac{\lambda}{2}(\nabla h)^2$ present in the equation to reflect the presence of lateral growth. The KPZ equation was solved with finite difference and RDTM, and the graphical plots for different values of $v$ , $\lambda$ and $\eta$ are presented in the Figure 1 and Figure 2.

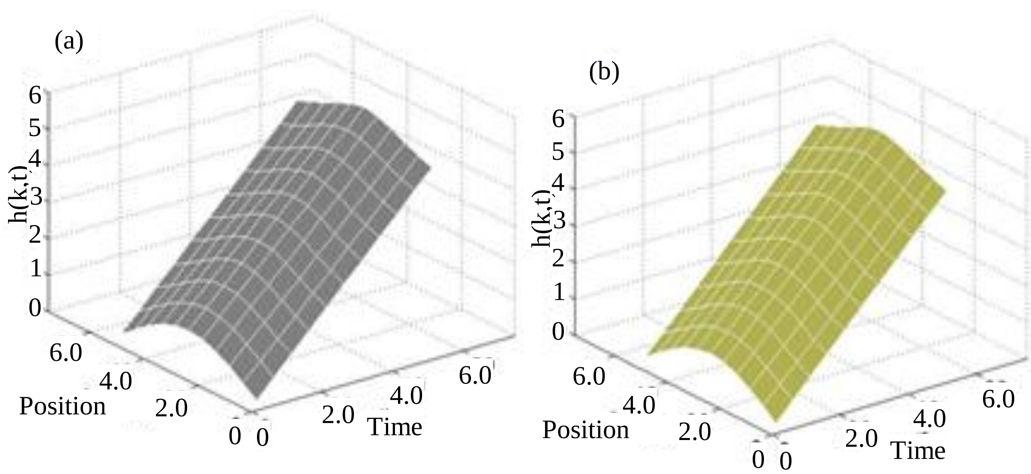


**Figure 1:** Numerical solution of the KPZ equation after 5000 time steps (5.0 seconds) for $v = 0.1$, $\lambda = -0.5$ and $\eta = +1$ with (a) Finite difference (b) RDTM.

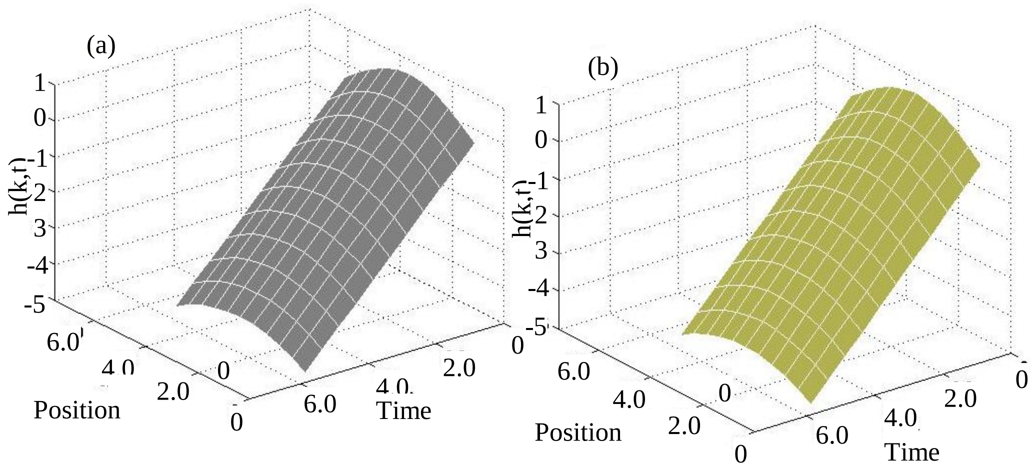


**Figure 2:** Numerical solution of the KPZ equation after 5000 time steps (5.0 seconds) for $v = 0.3$, $\lambda = 0.5$ and $\eta = -1$ with (a) Finite difference (b) RDTM.

The term of the KPZ containing $\upsilon$ is the surface relaxation term which tends to smooth the surface while the non-linear term containing $\lambda$ contributes lateral growth of surface. Figure 3-6 show the effect of increasing or decreasing the value of $\lambda$ which is associated to lateral growth (erosion).

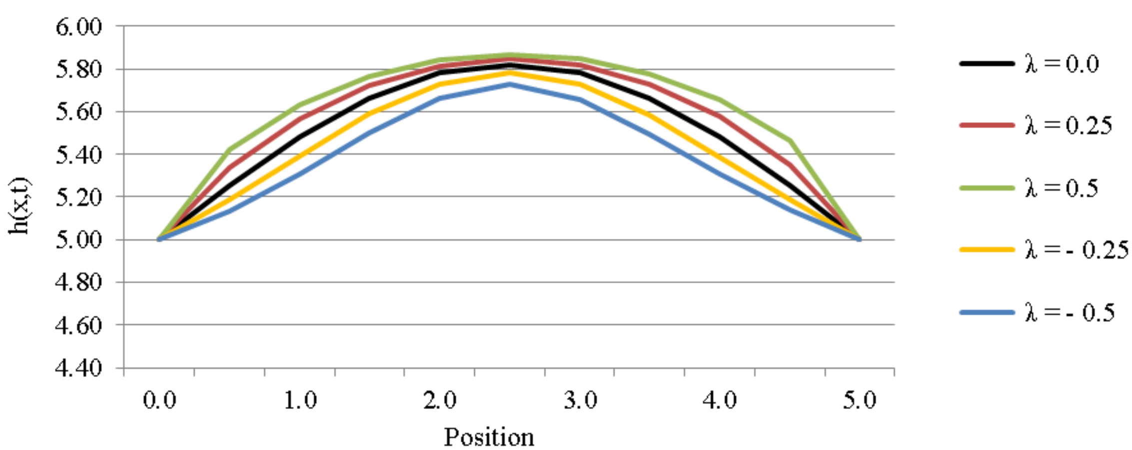


**Figure 3:** Effect of increasing $\lambda$ on surface lateral growth for $v = 0.1$ and $\eta = +1$.

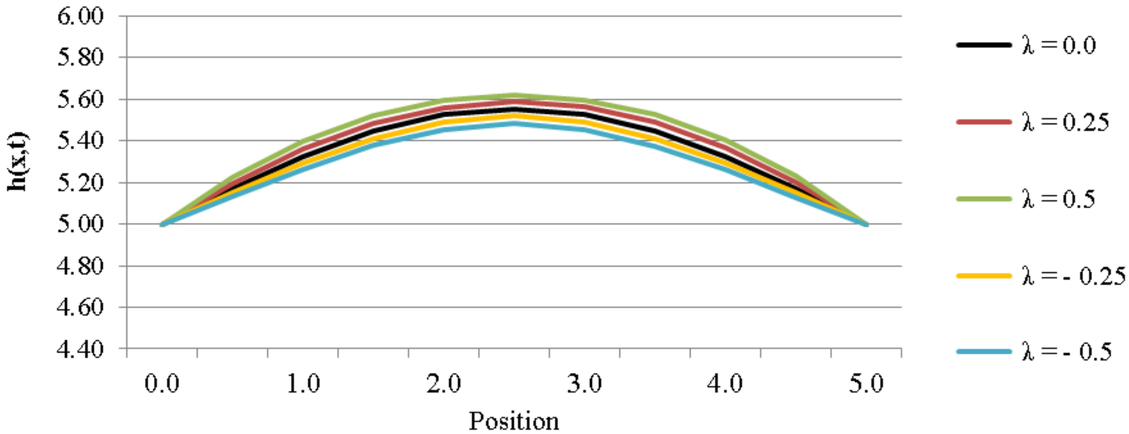


**Figure 4:** Effect of increasing $\lambda$ on surface lateral growth for $v = 0.3$ and $\eta = +1$.

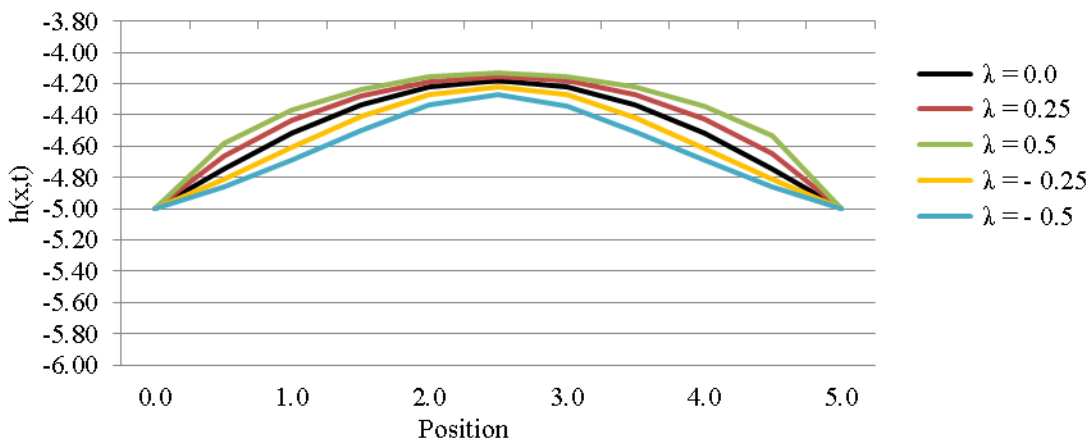


**Figure 5:** Effect of increasing $\lambda$ on surface lateral growth for $v = 0.1$ and $\eta = -1$.

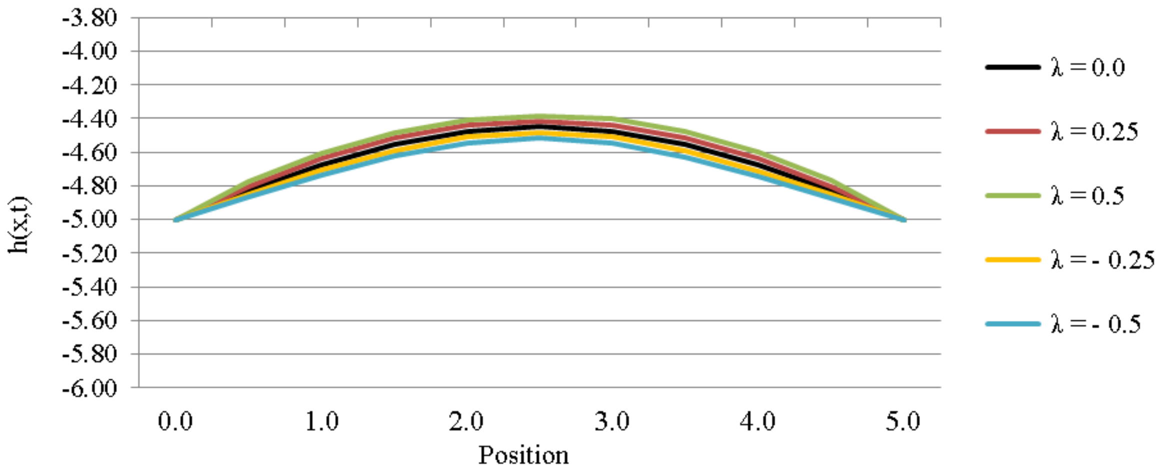


**Figure 6:** Effect of increasing $\lambda$ on surface lateral growth for $\nu = 0.3$ and $\eta = -1$.

The graphs in Figures 3–6 show the effect of the non-linear term of the KPZ equation. The surface was found to grow laterally for a positive increase in $\lambda$ thereby leading to increase in average surface height whereas a negative increase in $\lambda$ produces lateral erosion, leading to decrease in average surface height. The effect of increasing the surface tension term is also observed in comparing Figure 3 with Figure 4 and Figure 5 with Figure 6.

## 5. CONCLUSIONS

In this paper, we have successfully extended the reduced differential transform method to the stochastic Kadar-Parisi-Zhang equation. The solution obtained from RDTM was compared with the finite difference solution. RDTM gave comparable result to the finite difference solution for the Kadar-Parisi-Zhang equation and is proposed for solving other nonlinear partial differential equation.

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