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A Rank Test on Equality of Population Medians

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Abstract

The Kruskal-Wallis test is a non-parametric test for the equality of K population medians. The test statistic involved is a measure of the overall closeness of the K average ranks in the individual samples to the average rank in the combined sample. The resulting acceptance region of the test however may not be the smallest region with the required acceptance probability under the null hypothesis. Presently an alternative acceptance region is constructed such that it has the smallest size, apart from having the required acceptance probability. Compared to the Kruskal-Wallis test, the alternative test is found to have larger average power computed from the powers along the evenly chosen directions of deviation of the medians.

Keywords : non-parametric test ; population medians ; average power

1. INTRODUCTION

Let $X_{11},...,X_{n_1 1},...,X_{1K},...,X_{n_K K}$ be K independent random samples from continuous distributions with cumulative distribution functions $F\left(x-\theta_1\right),...,F\left(x-\theta_K\right)$, respectively where θ_j denotes a location parameter for the j-th population, frequently interpreted as the median or the treatment effect. We consider here the problem of testing the null hypothesis $H_0 : \theta_1 = ... = \theta_K$; that is, the hypothesis that there are no differences among the K population medians. The alternative hypothesis is $H_1 : \theta_j \neq \theta_k$ for at least one $j \neq k$.

Let $N = \sum_{j=1}^{K} n_j$ be the total number of observations in the combined sample. We first rank all N observations jointly, form least to greatest. Let r_{ij} denote the rank of X_{ij} that is $r_{ij} = \text{Rank}\left(X_{ij}\right)$ in the combined sample. For $j = 1,2, ..., K$, we set

$$R_j = \sum_{i=1}^{n_j} r_{ij}, \qquad \overline{R}_j = \frac{R_j}{n_j}, \qquad \overline{R}_{..} = \frac{N+1}{2}$$

where R_j is the sum of the ranks for the j-th treatment, $\overline{R}_j$ the average rank for the j-th treatment, and $\overline{R}_{..}$ the average rank in the joint ranking. A way to measure the overall closeness of the $\overline{R}_j$ to $\overline{R}_{..}$ is a weighted sum of the squared differences $\left(\overline{R}_j - \frac{N+1}{2}\right)^2$, for example, the Kruskal-Wallis statistic:

$$T = \frac{12}{N(N+1)} \sum_{j=1}^{K} n_j \left(\overline{R}_j - \frac{N+1}{2}\right)^2 \tag{1}$$

Since T is zero when the $\overline{R}_j$ are all equal and is large when there are substantial differences among the $\overline{R}_j$, the hypothesis is rejected for large values of T . For K = 2, the Kruskal-Wallis test reduces to the two-sided Wilcoxon test [1]. By squaring out $\left(\overline{R}_j - \frac{N+1}{2}\right)^2$ and replacing $\overline{R}_j$ by R_j. The statistic T can be rewritten as:

$$T = \left(\frac{12}{N\left(N+1\right)} \sum_{j=1}^{K} \frac{R_j^2}{n_j}\right) - 3\left(N+1\right)$$

The null distribution of T can be obtained by using the fact that under H_0, all $N!/\prod_{j=1}^{K} n_j!$ assignments of n_1 ranks to the treatment 1 observations, n_2 ranks to the treatment 2 observations, …, n_K ranks to the treatment K observations, are equally likely. However, this method is computational difficult even K is small [2] provided the upper 10% point of the exact probability distribution of the Kruskal-Wallis test statistic for $K = 3$ samples with $\max(n_1, n_2, n_3) \le 6$, also $n_1 = n_2 = n_3 = 7$ and 8; $K = 4$ samples with $(n_1, n_2, n_3, n_4) \le 4$; and $K = 5$ samples with $(n_1, n_2, n_3, n_4, n_5) \le 3$.

Many approximate distributions of the statistic T under the null hypothesis were proposed because of the computational difficulty for computing the exact null distribution of the T statistic. Kruskal [3] showed that under the null hypothesis, the statistic T has a limiting chi-square distribution with $K - 1$ degrees of freedom if $\text{minimum}(n_1,...,n_K) \to \infty$, with $n_j / N \to \lambda_j, 0 < \lambda_j < 1$, for $j = 1,2, ..., K$ For finite samples, the approximation based on this asymptotic result is in general conservative; that is, it indicates upper-tail probabilities which are larger than the true ones. An alternative simple approximation is given by Wallace [4] ; the B_2-III approximation. This approximation is generally closer than the preceding one, but it tends to be anticonservative unless the sample sizes are quite disparate, in which case it becomes conservative also. There are several other approximations [5,6].

The acceptance region of the Kruskal Wallis test may not be the smallest region with the probability $1-\alpha$ under $\mathrm{H_o}$. Thus if we can find an alternative test of which the acceptance region is smallest and yet having the required probability $1-\alpha$ under $\mathrm{H_o}$, then it is likely that the alternative test may be able to achieve larger power under the alternative hypothesis.

In Section 2, we find an approximate multivariate quadratic-normal distribution for $\mathbf{R} = (R_1, R_2, ..., R_K)$ and use the underlying random variables which have independent standard normal distributions to form the smallest acceptance region with the required acceptance probability.

In Section 3, we compare the powers of the Kruskal Wallis test and the alternative test under different types of distribution of the X_{ij}. It is found that the alternative test has larger average power computed from the powers along the evenly chosen directions of deviation of the medians.

2. AN ALTERNATIVE TEST FOR THE EQUALITY OF POPULATION MEDIANS

We note the Kruskal-Wallis test statistic T is a function of the random variables, $R_1, R_2, ..., R_K$ which are correlated and non-normally distributed. Therefore the acceptance region given by the values of $(R_1, R_2, ..., R_K)$ of which T is not larger than a constant T_α may not be the smallest region with the required acceptance probability.

Presently we find an approximate multivariate non-normal distribution for $\mathbf{R}$ in terms of a set of uncorrelated random variables $z_1, z_2, ..., z_{K-1}$ having the standard normal distributions, and propose an alternative test of which the acceptance region is given by $z_1^2 + z_2^2 + ... + z_{K-1}^2 \le \chi^2_{K-1,\alpha}$ where $\chi^2_{K-1,\alpha}$ is the $100(1-\alpha)\%$ point of the chi-square distribution with K–1 degrees of freedom. The bell shape of the normal distributions implies that the acceptance region will be the smallest region with the required probability $1-\alpha$.

We may use the following procedure to find an approximate multivariate non-normal distribution for $\mathbf{R}$:

i. Generate M values of $\mathbf{R}$ using a chosen common continuous distribution for the X_{ij}. As the null distribution of $\mathbf{R}$ does not depend on the common distribution of the X_{ij}, we may choose the common continuous distribution to be the standard normal distribution.

ii. Compute the sample moments

$$M_{jk}^{(k_1)(k_2)} = \frac{1}{M}\sum_{m=1}^{M} \overline{R}_{mj}^{k_1} \overline{R}_{mk}^{k_2} \qquad j,k = 1,2,\ldots,K;\ k_1 \ge 0,\ k_2 \ge 0,\ 0 \le k_1 + k_2 \le 2$$

where $(R_{m1}, R_{m2}, \ldots, R_{mK})$ is the m-th generated value of $\mathbf{R}$, $\overline{R}_{mj} = R_{mj} - \overline{R}_j$ and $\overline{R}_j = \frac{1}{M}\sum_{m=1}^{M} R_{mj}$.

iii. Use the results obtained in Step (ii) to compute the sample variance-covariance matrix of $\mathbf{R}$.

iv. Find the K eigenvectors of the sample variance-covariance matrix obtained in Step (iii), and use the K-1 eigenvectors with non-negligible eigenvalues to form the matrix $\boldsymbol{B}$.

v. Compute $(S_{m1}, S_{m2}, \ldots, S_{m(K-1)})^T = \boldsymbol{B}^T (R_{m1}, R_{m2}, \ldots, R_{mK})^T$.

vi. Compute $M_j^{(k)} = \frac{1}{M}\sum_{m=1}^{M} S_{mj}^k$, $\quad k = 2,3,4; \quad j = 1,2,\ldots,K-1$

vii. Find $\underset{\sim}{\lambda}^{(j)} = \left(\lambda_1^{(j)}, \lambda_1^{(j)}, \lambda_3^{(j)}\right)$ such that $E(\varepsilon_j^k) = M_j^{(k)}$, $k = 2,3,4$ where

$$\varepsilon_j = \begin{cases} \lambda_1^{(j)} z_j + \lambda_2^{(j)} \left(z_j^2 - \dfrac{1-\lambda_3^{(j)}}{2} \right), & for \;\; z_j \geq 0 \\ \lambda_1^{(j)} z_j + \lambda_2^{(j)} \left(\lambda_3^{(j)} z_j^2 - \dfrac{1-\lambda_3^{(j)}}{2} \right), & for \;\; z_j < 0 \end{cases} \tag{2}$$

The random variable ε_j in Step (vii) is said to have a quadratic-normal distribution with parameters 0 and $\boldsymbol{\lambda}^{(j)}$ [7] while the vector $\mathbf{R} = (\mathrm{R}_1, \mathrm{R}_2, ..., \mathrm{R}_\mathrm{K})$ is said to have a (K-1)-dimensional multivariate quadratic-normal distribution with estimated parameters $(\overline{\mathrm{R}}_1, \overline{\mathrm{R}}_2, ..., \overline{\mathrm{R}}_\mathrm{K})$, $\mathbf{B}$, and $\boldsymbol{\lambda}^{(i)}$, i = 1, 2, …, K-1 [8].

3. POWERS OF THE ALTERNATIVE TEST

In this section, we use simulation to estimate the powers of the alternative test under the following three possible distributions of the X_{ij}

i. $X_{ij} \sim N(\theta_j, 1)$

ii. $X_{ij} \sim$ Uniform distribution over $[\theta_j, \theta_j + 1]$

iii. $X_{ij} = \theta_j + X_j^{(1)} + X_j^{(2)}$
where $X_j^{(1)} \sim$ Uniform distribution over [-0.5, 0.5] and $X_j^{(2)} \sim N(0,1)$

For a given value of $(\theta_1, \theta_2, \ldots, \theta_K)$ we may estimate the powers under a given type of distribution of the X_{ij} by first generating M^* values of $\mathbf{X} = (X_{11}, ..., X_{n_1 1}, ..., X_{1K}, ..., X_{n_K K})$. For a generated value of $\mathbf{X}$, we find the corresponding values of $\mathbf{R}$, $\mathbf{S}^T = \mathbf{B}^T \mathbf{R}^T$ and $\boldsymbol{\lambda}^{(j)}$, j = 1, 2, …, K–1.

By solving Equations (2) with ε_j replaced by the computed S_j, we get the value of z_j, j = 1, 2, …, K-1. If $\sum_{j=1}^{K-1} z_j^2 > \chi^2_{K-1,\alpha}$, then we get a rejection result. The proportion of the M^* generated values of $\mathbf{X}$ which lead to rejection results is then an estimate of the power of the test evaluated at $(\theta_1, \theta_2, \ldots, \theta_K)$.

To compare the power functions of the tests, we choose a number of values of the radial distance ρ in the K dimensional polar coordinates system, and for each chosen value of the radial distance, we choose evenly M_β values of the vector $(\beta_1, \beta_2, \ldots, \beta_{K-1})$ of the polar angles. From the value of $(\rho, \beta_1, \beta_2, \ldots, \beta_{K-1})$, we determine the value of $(\theta_1, \theta_2, \ldots, \theta_K)$ and estimate the corresponding powers of the tests. For each value of ρ, we then find the average of the powers over the M_β evenly chosen values of the vector of the polar angles. Some results for average powers for the case when K=3 are shown in Tables 3.1 and 3.2.

Table 3.1: Average Powers of the Tests (K=3, $n_1=n_2=n_3=3$,α=0.05, M=50,000, M*=10,000 , M_β =81)

P	Kruskal-Wallis Tests			Alternative Test		
	Normal	Uniform	Mixed	Normal	Uniform	Mixed
0	0.050285	0.0498□7	0.049995	0.050119	0.049857	0.049998
0.5	0.072399	0.281719	0.070769	0.073219	0.284216	0.07126
1	0.143315	0.60751	0.136123	0.145281	0.61794	0.138358
1.5	0.254288	0.739375	0.239963	0.25541	0.766064	0.241651
2	0.373773	0.81271	0.354752	0.375146	0.844965	0.356468
2.5	0.477936	0.858073	0.459284	0.482183	0.89261	0.46207
3	0.559098	0.886293	0.542122	0.566735	0.918815	0.54781
3.5	0.620504	0.904128	0.604965	0.632441	0.933756	0.615675
4	0.667812	0.915767	0.654838	0.68412	0.942733	0.669419
4.5	0.705969	0.926141	0.692505	0.726409	0.949969	0.711978
5	0.736262	0.934433	0.725464	0.76017	0.955236	0.747995

Table 3.2: Average Powers of the Tests (K=3, $n_1=2$,$n_2=5$,$n_3=8$, α=0.05, M=50,000, M*=10,000 , M_β =81)

P	Kruskal-Wallis Tests			Alternative Test		
	Normal	Uniform	Mixed	Normal	Uniform	Mixed
0	0.048917	0.049041	0.048989	0.048607	0.048479	0.048575
0.5	0.090201	0.453569	0.08736	0.088452	0.469756	0.085716
1	0.224562	0.69524	0.210353	0.222264	0.746101	0.207654
1.5	0.399486	0.791251	0.380105	0.406342	0.848244	0.385317
2	0.534953	0.847688	0.518285	0.559169	0.904767	0.539495
2.5	0.615925	0.882475	0.603649	0.652763	0.937301	0.639084
3	0.666621	0.905001	0.656399	0.711836	0.95556	0.699816
3.5	0.704638	0.921752	0.695489	0.75374	0.965957	0.743068
4	0.736279	0.933859	0.726521	0.788341	0.971599	0.778041
4.5	0.76293	0.943414	0.754138	0.816983	0.976267	0.806938
5	0.786169	0.951927	0.777465	0.841758	0.980242	0.832007

Tables 3.1 and 3.2 show that for sufficiently large value of ρ, the average power of the alternative test is always larger than that of the Kruskal-Wallis test, and as the sample sizes become disparate and the distributions of the X_{ij} deviate from normality, the difference between the average powers of the two tests becomes more obvious.

4. CONCLUSION

For the values of K and the n_j chosen so far, it is possible to approximate the null distribution of the vector formed by the sums of the ranks within sample. When the values of K and the n_j are such that the multivariate quadratic-normal distribution is not suitable, we may use other multivariate non-normal distributions formed by replacing the quadratic functions in Equation (2) by other nonlinear functions and apply the method given in Section 2. It is likely that for other values of K, the situation with disparate sample sizes and non-normal distributions of the X_{ij} will continue to be one in which the alternative test will perform better.

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