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New Inequality for L-Lipschitzian Functions and Applications

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Abstract: This article gives a really vital and curiously inequality on Jain-Saraswat's functional discrimination in terms of the Hellinger discrimination and Bhattacharya discrimination by taking into thought L-Lipschitzian functions. Encourage, we outlined a few vital results by utilizing the inferred inequality with numerical confirmation.

Index terms: L- Lipschitzian functions, New information inequality, Binomial and Poisson probability distributions, Hellinger discrimination, Bhattacharya discrimination.

Mathematics Subject Classification: Primary 94A17, Secondary 26D15.

1. Introduction

Discrimination measures are used in measuring the distance or affinity among finite number of probability distributions (both discrete and continuous). Actually, these are for quantifying the dissimilarity among probability distributions.Some researchers, like: Csiszar (1966), Bregman (1967), Burbea- Rao (1982), Lin- Wong (1995) and Jain- Saraswat's (2013) etc. took a deep study on functional discrimination measures. After putting a fitting work in these functional discrimination measures, a few popular discrimination measures can be gotten, like: Kullback Leibler discrimination measure, J- discrimination measure, Arithmetic geometric mean discrimination measure, Jensen Shannon mean discrimination measure, Bhattacharya discrimination measure and many more.

As of late, discrimination measures are being utilized in a few areas, like: science [36], guess of likelihood conveyances [14, 4], fetched- delicate classification for therapeutic conclusion [39], choice making [29], color picture division [37], 3D picture division and word arrangement [41], financial matters and political science [5, 43], attractive reverberation picture investigation [44], turbulence stream [7], examination of possibility tables [35], design acknowledgment [30] etc.

Let $\Upsilon_m = \{\Theta = (\theta_1, \theta_2, \theta_3, \ldots, \theta_m) : \theta_p > 0, \sum_{p=1}^{m} \theta_p = 1\}$, $m \geq 2$ be the set of all complete finite discrete probability distributions. If we take $\theta_p \geq 0$ for some $p = 1,2,3\ldots, m$, then we have to suppose that $0g(0) =$

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$0g\left(\frac{0}{0}\right) = 0$.

Csiszar presented a functional discrimination measure [1, 9], which is brodly utilized due to its compact nature, it is given by

$$\Lambda g(\Theta, \Phi) = \sum_{p=1}^{m} \phi_p g\left(\frac{\theta_p}{\phi_p}\right), \tag{1}$$

where $g: (0, \infty) \to R$ (set of real no.) is real, continuous, and convex function and $\Theta = (\theta_1, \theta_2, \ldots, \theta_m), \Phi = (\phi_1, \phi_2, \ldots, \phi_m) \in \Upsilon_m$, where θ_p and ϕ_p are probability mass functions.

Essentially, Jain and Saraswat gave the taking after functional discrimination measure [27],

$$\Xi_g(\Theta, \Phi) = \sum_{p=1}^{m} \phi_p g\left(\frac{\theta_p + \phi_p}{2\phi_p}\right). \tag{2}$$

$\Lambda g(\Theta, \Phi)$ and $\Xi_g(\Theta, \Phi)$ are common separate measures from a genuine likelihood dissemination Θ to an subjective likelihood dissemination Φ. Really Θ speaks to perceptions, though Φ speaks to an guess of Θ. Numerous separation measures can be gotten by employing an appropriate convex function in $\Lambda g(\Theta, \Phi)$ and $\Xi_g(\Theta, \Phi)$. The properties of $\Xi_g(\Theta, \Phi)$ and their proofs can be seen in literature [27] and several information inequalities on $\Xi_g(\Theta, \Phi)$ and their applications can be seen in the articles [8] and ([17]- [23]).

Definition 1.1 ***Convex function****: A function $g(y)$ is said to be convex over an interval (a, b) if for every $y_1, y_2 \in (a, b)$ and $0 \leq \xi \leq 1$, we have*

$$g[\xi y_1 + (1 - \xi) y_2] \leq \xi g(y_1) + (1 - \xi) g(y_2), \tag{3}$$

and said to be strictly convex if equality does not hold only if $\xi \neq 0$ or $\xi \neq 1$.

In generalized way, we can write

$$g\left[\sum_{p=1}^{m} \xi_p y_p\right] \leq \sum_{p=1}^{m} \xi_p g(y_p),$$

for all $y_p \in (a, b)$ and $\xi_p \geq 0$ with $\sum_{p=1}^{m} \xi_p = 1$.

In hypothesis of imbalances, Convex functions play an imperative part. In case disparity (3) holds in reversed direction at that point, g is said to be concave. Convex functions have wide applications in immaculate and connected science, material science and other characteristic sciences. As of late numerous generalizations and expansions have been made for the convexity, like s- convexity [3], strong convexity [45], preinvexity [31], GA- convexity [46], GG- convexity [32], and others.

Definition 1.2 ***L- Lipschitzian function****: A function g is called L- Lipschitzian over a set S with respect to a norm $\|.\|$ if for all $\omega, \sigma \in S$, we have*

$$|g(\omega) - g(\sigma)| \leq L\|\omega - \sigma\|.$$

Some people will equivalently say g is Lipschitzian continuous with Lipschitzian constant L. Intuitively, L is a measure of how fast the function can change.

2. A New Inequality on Functional Discrimination $\Xi_g(\Theta, \Phi)$

In this segment, we'll determine a original information inequality on functional discrimination measure $\Xi_g(\Theta, \Phi)$ by taking into thought L- Lipschitzian functions. We first start with the following theorem, given by [11].

Theorem 2.1 *Let Ω, η be two normed linear spaces with the norms $\|.\|$ and $|.|$ respectively. If $g: \Omega \to \eta$ is L- Lipschitzian, then $\forall \omega_p \in \Omega, \theta_p \geq 0$ with $\sum_{p=1}^{m} \theta_p = 1, p = 1,2,3, \ldots, m$, we have*

$$\left|g\left(\sum_{p=1}^{m} \theta_p \omega_p\right) - \sum_{p=1}^{m} \theta_p g\left(\omega_p\right)\right| \le L \sum_{p=1}^{m} \theta_p\left(1-\theta_p\right) \sum_{l=1}^{m-1} \|\Delta\omega_l\|, \tag{4}$$

where $\Delta\omega_l = \omega_{l+1} - \omega_l$ is the forward difference.

By utilizing the inequality (4), a new inequality on $\Xi_g(\Theta,\Phi)$ can be inferred in terms of the well known Hellinger discrimination and Bhattacharya discrimination.

Theorem 2.2 *Let $g:[\mu,\zeta] \subset (0,\infty) \to (-\infty,\infty)$ be L- Lipschitzian and differentiable convex function with bounded derivative, defined on $[\mu,\zeta]$ with $0<\mu\le 1\le \zeta<\infty, \mu\ne\zeta$. For $\Theta,\Phi \in \Upsilon_m$, we have*

$$\Xi_g(\Theta,\Phi) \le \left[h(\Theta,\Phi) + B(\Theta,\Phi) - \sum_{p=1}^{m} \phi_p^2\right]\left[\sum_{l=1}^{m-1}\left|\frac{\theta_{l+1}}{\phi_{l+1}} - \frac{\theta_l}{\phi_l}\right|\right] \sup_{y\in[\mu,\zeta]} |g'(y)|, \tag{5}$$

where

$$h(\Theta,\Phi) = \sum_{p=1}^{m} \frac{\left(\sqrt{\theta_p}-\sqrt{\phi_p}\right)^2}{2} \tag{6}$$

is the Hellinger Discrimination or Kolmogorov's discrimination [15] and

$$B(\Theta,\Phi) = \sum_{p=1}^{m} \sqrt{\theta_p \phi_p} \tag{7}$$

is the Bhattacharya discrimination [6] and $\Xi_g(\Theta,\Phi)$ is defined in equation (2).

Proof: First replace θ_p with ϕ_p for $p=1,2,\dots,m$ in inequality (4), we have

$$\left|g\left(\sum_{p=1}^{m} \phi_p \omega_p\right) - \sum_{p=1}^{m} \phi_p g\left(\omega_p\right)\right| \le L \sum_{p=1}^{m} \phi_p\left(1-\phi_p\right) \sum_{l=1}^{m-1} |\omega_{l+1}-\omega_l|.$$

Now put $\omega_p = \frac{\theta_p+\phi_p}{2q_p}$ and by considering $\sum_{p=1}^{m}\theta_p = \sum_{p=1}^{m}\phi_p = 1$, we have

$$\left|g(1) - \sum_{p=1}^{m} \phi_p g\left(\frac{\theta_p+\phi_p}{2q_p}\right)\right| \le L\left[1-\sum_{p=1}^{m}\phi_p^2\right]\left[\sum_{l=1}^{m-1}\left|\frac{\theta_{l+1}}{\phi_{l+1}} - \frac{\theta_l}{\phi_l}\right|\right].$$

If function g is normalized, p.e., $g(1)=0$ and also has bounded derivative in interval $[\mu,\zeta]$, then by the definition of 1.2, we have

$$\left|-\Xi_g(\Theta,\Phi)\right| \le \sup_{y\in[\mu,\zeta]}|g'(y)|\left[\sum_{p=1}^{m}\frac{\theta_p+\phi_p}{2} - \sum_{p=1}^{m}\phi_p^2\right]\left[\sum_{l=1}^{m-1}\left|\frac{\theta_{l+1}}{\phi_{l+1}} - \frac{\theta_l}{\phi_l}\right|\right].$$

$$\Rightarrow \Xi_g(\Theta,\Phi) \le \sup_{y\in[\mu,\zeta]}|g'(y)|\left[h(\Theta,\Phi)+B(\Theta,\Phi)-\sum_{p=1}^{m}\phi_p^2\right]\left[\sum_{l=1}^{m-1}\left|\frac{\theta_{l+1}}{\phi_{l+1}} - \frac{\theta_l}{\phi_l}\right|\right].$$

Which is a required result.

Since

$$\sum_{p=1}^{m} \frac{\theta_p+\phi_p}{2} - \sum_{p=1}^{m} \phi_p^2 = \sum_{p=1}^{m} \frac{\theta_p+\phi_p-2\phi_p^2}{2} = \sum_{p=1}^{m} \frac{\theta_p+\phi_p-2\sqrt{\theta_p\phi_p}+2\sqrt{\theta_p\phi_p}-2\phi_p^2}{2}$$

$$= \sum_{p=1}^{m} \frac{\left(\sqrt{\theta_p}-\sqrt{\phi_p}\right)^2}{2} + \sum_{p=1}^{m} \sqrt{\theta_p\phi_p} - \sum_{p=1}^{m} \phi_p^2$$

$$= h(\Theta,\Phi) + B(\Theta,\Phi) - \sum_{p=1}^{m} \phi_p^2$$

Note: The later applications of the Hellinger discrimination (in information examination) and Bhattacharya discrimination (in space reconnaissance) can be cited within the articles [2] and [16], individually.

3. Main Results

By utilizing the determined inequality (5), presently we'll assess a few extraordinary results among distinctive discriminations.

Result 3.1 *For $\Theta, \Phi \in \Upsilon_m$ and $0 < \mu \leq 1 \leq \zeta < \infty, \mu \neq \zeta$, we have*

a. If $0 < \mu < 1$, then

$$\Delta(\Theta,\Phi) \leq \left[h(\Theta,\Phi) + B(\Theta,\Phi) - \sum_{p=1}^{m} \phi_p^2\right]\left[\sum_{l=1}^{m-1} \left|\frac{\theta_{l+1}}{\phi_{l+1}} - \frac{\theta_l}{\phi_l}\right|\right]\left[\frac{\zeta^2-\mu^2}{\mu^2\zeta^2} + \left|\frac{\zeta^2+\mu^2}{\mu^2\zeta^2} - 2\right|\right]. \tag{8}$$

b. If $\mu = 1$, then

$$\Delta(\Theta,\Phi) \leq 2\left[h(\Theta,\Phi) + B(\Theta,\Phi) - \sum_{p=1}^{m} \phi_p^2\right]\left[\sum_{l=1}^{m-1} \left|\frac{\theta_{l+1}}{\phi_{l+1}} - \frac{\theta_l}{\phi_l}\right|\right]\left(\frac{\zeta^2-1}{\zeta^2}\right), \tag{9}$$

where $h(\Theta,\Phi)$, $B(\Theta,\Phi)$ are characterized in conditions (6) and (7) separately and $\Delta(\Theta,\Phi)$ is characterized underneath within the confirmation.

Proof: Let

$$g(y) = \frac{(y-1)^2}{y}, y \in R_+, g(1) = 0, g'(y) = \frac{y^2-1}{y^2} \text{ and } g''(y) = \frac{2}{y^3}.$$

Since $g''(y) > 0 \ \forall \ y > 0$ and $g(1) = 0$, so $g(y)$ is strictly convex and normalized function respectively.

For this function, from equation (2), we have

$$\Xi_g(\Theta,\Phi) = \frac{1}{2}\sum_{p=1}^{m} \frac{(\theta_p-\phi_p)^2}{\theta_p+\phi_p} = \frac{1}{2}\Delta(\Theta,\Phi), \tag{10}$$

where $\Delta(\Theta,\Phi)$ is the famous Triangular discrimination [10].

Now, let $g(y) = |g'(y)| = \left|\frac{y^2-1}{y^2}\right| = \begin{cases} -\frac{y^2-1}{y^2} & \text{if } 0 < y < 1 \\ \frac{y^2-1}{y^2} & \text{if } 1 \leq y < \infty \end{cases}$, and

$$g'(y) = \begin{cases} -\dfrac{2}{y^3} & \text{if } 0 < y < 1 \\ \dfrac{2}{y^3} & \text{if } 1 \leq y < \infty \end{cases}.$$

It is clear that $g'(y) < 0$ in $(0,1)$ and > 0 in $(1,\infty)$, p.e., $g(y)$ is strictly decreasing in $(0,1)$ and strictly increasing in $(1,\infty)$, so

$$\sup_{y\in[\mu,\zeta]}|g'(y)| = \sup_{y\in[\mu,\zeta]} g(y) = \begin{cases} max[|g'(\mu)|,|g'(\zeta)|] = \frac{|g'(\mu)|+|g'(\zeta)|+||g'(\mu)|-|g'(\zeta)||}{2} & \text{if } 0<\mu<1 \\ |g'(\zeta)| & \text{if } \mu = 1 \end{cases}$$

$$= \begin{cases} \frac{1}{2}\left[\frac{\zeta^2-\mu^2}{\mu^2\zeta^2} + \left|\frac{\zeta^2+\mu^2}{\mu^2\zeta^2} - 2\right|\right] & \text{if } 0<\mu<1 \\ \frac{(\zeta+1)(\zeta-1)}{\zeta^2} & \text{if } \mu = 1 \end{cases}. \tag{11}$$

The results (8) and (9) can be gotten by putting the values from the equations (10) and (11) into the inequality (5).

Result 3.2 *For* $\Theta,\Phi \in \Upsilon_m$ *and* $0<\mu\le 1\le\zeta<\infty, \mu\ne\zeta$, *we have*

$$g(\Phi,\Theta) \le \frac{1}{\mu}\left[h(\Theta,\Phi) + B(\Theta,\Phi) - \sum_{p=1}^{m}\phi_p^2\right]\left[\sum_{l=1}^{m-1}\left|\frac{\theta_{l+1}}{\phi_{l+1}} - \frac{\theta_l}{\phi_l}\right|\right], \tag{12}$$

where $g(\Theta,\Phi)$ is defined below in the proof.

Proof: Let

$$g(y) = -\log y, y\in R_+, g(1)=0, g'(y) = -\frac{1}{y} \text{ and } g''(y) = \frac{1}{y^2}.$$

Since $g''(y)>0\ \forall\ y>0$ and $g(1)=0$, so $g(y)$ is strictly convex and normalized function respectively.

For this function, from equation (2), we have

$$\Xi_g(\Theta,\Phi) = \sum_{p=1}^{m}\phi_p\log\left(\frac{2\phi_p}{\theta_p+\phi_p}\right) = g(\Phi,\Theta), \tag{13}$$

where $g(\Phi,\Theta)$ is the adjoint of the Relative JS discrimination $g(\Theta,\Phi)$ [40].

Now, let $g(y) = |g'(y)| = \left|-\frac{1}{y}\right| = \frac{1}{y}$, and $g'(y) = -\frac{1}{y^2} < 0$.

We can clearly see that $g(y)$ is always strictly decreasing in $(0,\infty)$, so

$$\sup_{y\in[\mu,\zeta]}|g'(y)| = \sup_{y\in[\mu,\zeta]} g(y) = g(\mu) = \frac{1}{\mu}. \tag{14}$$

The result (12) can be obtained by putting the values of the equations (13) and (14) into the inequality (5).

Result 3.3 *For* $\Theta,\Phi \in \Upsilon_m$ *and* $0<\mu\le 1\le\zeta<\infty, \mu\ne\zeta$, *we have*

a. If $0<\mu\le\frac{1}{e}$, then

$$G(\Phi,\Theta) \le \left[h(\Theta,\Phi) + B(\Theta,\Phi) - \sum_{p=1}^{m}\phi_p^2\right]\left[\sum_{l=1}^{m-1}\left|\frac{\theta_{l+1}}{\phi_{l+1}} - \frac{\theta_l}{\phi_l}\right|\right]\left[\log\sqrt{\frac{\zeta}{\mu}} + \left|1+\log\sqrt{\mu\zeta}\right|\right]. \tag{15}$$

b. If $\frac{1}{e}<\mu\le 1$, then

$$G(\Phi,\Theta) \le \left[h(\Theta,\Phi) + B(\Theta,\Phi) - \sum_{p=1}^{m}\phi_p^2\right]\left[\sum_{l=1}^{m-1}\left|\frac{\theta_{l+1}}{\phi_{l+1}} - \frac{\theta_l}{\phi_l}\right|\right](1+\log\mu), \tag{16}$$

where $G(\Theta,\Phi)$ is characterized underneath within the proof.

Proof: Let

$$g(y) = y\log y, y\in R_+, g(1)=0, g'(y) = 1+\log y \text{ and } g''(y) = \frac{1}{y}.$$

Since $g''(y)>0\ \forall\ y>0$ and $g(1)=0$, so $g(y)$ is strictly convex and normalized function respectively.

For this function, from equation (2), we have

$$\Xi_g(\Theta,\Phi) = \sum_{p=1}^{m}\frac{\theta_p+\phi_p}{2}\log\left(\frac{\theta_p+\phi_p}{2\phi_p}\right) = G(\Phi,\Theta), \tag{17}$$

where $G(\Phi,\Theta)$ is the adjoint of the Relative AG discrimination $G(\Theta,\Phi)$ [42].

Now, let $g(y) = |g'(y)| = |1+\log y| = \begin{cases} -1-\log y & \text{if } 0 < y \le \frac{1}{e} \\ 1+\log y & \text{if } \frac{1}{e} < y < \infty \end{cases}$, and

$$g'(y) = \begin{cases} -\dfrac{1}{y} & \text{if } 0 < y \le \dfrac{1}{e} \\ \dfrac{1}{y} & \text{if } \dfrac{1}{e} < y < \infty \end{cases}.$$

Since $g'(y) < 0$ in $\left(0,\frac{1}{e}\right)$ and > 0 in $\left(\frac{1}{e},\infty\right)$, p.e., $g(y)$ is strictly decreasing in $\left(0,\frac{1}{e}\right)$ and strictly increasing in $\left(\frac{1}{e},\infty\right)$, therefore

$$\sup_{y\in[\mu,\zeta]} |g'(y)| = \sup_{y\in[\mu,\zeta]} g(y) =$$

$$\begin{cases} max[|g'(\mu)|,|g'(\zeta)|] = \left[\log\sqrt{\frac{\zeta}{\mu}} + \left|1+\log\sqrt{\mu\zeta}\right|\right] & \text{if } 0 < \mu \le \frac{1}{e} \\ |g'(\zeta)| = 1+\log\zeta & \text{if } \frac{1}{e} < \mu \le 1. \end{cases} \tag{18}$$

The results (15) and (16) can be gotten by putting the values of the equations (17) and (18) into the inequality (5).

In a comparative way, we get the taking after results for diverse convex functions. Subtle elements are excluded.

Result 3.4 *For* $g(y) = (y-1)logy$, $\Theta,\Phi \in \Upsilon_m$ *and* $0 < \mu \le 1 \le \zeta < \infty, \mu \ne \zeta$, *we have*

a. If $0 < \mu < 1$, then

$$J_R(\Theta,\Phi) \le 2\left[h(\Theta,\Phi) + B(\Theta,\Phi) - \sum_{p=1}^{m} \phi_p^2\right]\left[\sum_{l=1}^{m-1} \left|\frac{\theta_{l+1}}{\phi_{l+1}} - \frac{\theta_l}{\phi_l}\right|\right]\left[\log\sqrt{\frac{\zeta}{\mu}} + \frac{\zeta-\mu}{2\mu\zeta} + \left|\frac{\zeta+\mu}{2\mu\zeta} - \log e\sqrt{\mu\zeta}\right|\right]. \tag{19}$$

b. If $\mu = 1$, then

$$J_R(\Theta,\Phi) \le 2\left[h(\Theta,\Phi) + B(\Theta,\Phi) - \sum_{p=1}^{m} \phi_p^2\right]\left[\sum_{l=1}^{m-1} \left|\frac{\theta_{l+1}}{\phi_{l+1}} - \frac{\theta_l}{\phi_l}\right|\right]\left(\log e\zeta - \frac{1}{\zeta}\right), \tag{20}$$

where

$$\Xi_g(\Theta,\Phi) = \frac{1}{2}\sum_{p=1}^{m} \left(\theta_p - \phi_p\right)\log\left(\frac{\theta_p+\phi_p}{2\phi_p}\right) = \frac{1}{2}J_R(\Theta,\Phi). \tag{21}$$

$J_R(\Theta,\Phi)$ is known as the Relative J- discrimination [12].

Result 3.5 *For* $g(y) = (y-1)^2$, $\Theta,\Phi \in \Upsilon_m$ *and* $0 < \mu \le 1 \le \zeta < \infty, \mu \ne \zeta$, *we have*

a. If $0 < \mu < 1$, then

$$\chi^2(\Theta,\Phi) \le 4\left[h(\Theta,\Phi) + B(\Theta,\Phi) - \sum_{p=1}^{m} \phi_p^2\right]\left[\sum_{l=1}^{m-1} \left|\frac{\theta_{l+1}}{\phi_{l+1}} - \frac{\theta_l}{\phi_l}\right|\right][(\zeta-\mu) + |2-(\zeta+\mu)|]. \tag{22}$$

b. If $\mu = 1$, then

$$\chi^2(\Theta,\Phi) \le 8\left[h(\Theta,\Phi) + B(\Theta,\Phi) - \sum_{p=1}^{m} \phi_p^2\right]\left[\sum_{l=1}^{m-1} \left|\frac{\theta_{l+1}}{\phi_{l+1}} - \frac{\theta_l}{\phi_l}\right|\right](\zeta-1), \tag{23}$$

where

$$\Xi_g(\Theta,\Phi) = \frac{1}{4}\sum_{p=1}^{m} \frac{(\theta_p-\phi_p)^2}{\phi_p} = \frac{1}{4}\chi^2(\Theta,\Phi). \tag{24}$$

$\chi^2(\Theta,\Phi)$ is designated as the Chi- square discrimination or Pearson discrimination [38].

Result 3.6 *For* $g(y) = |y-1|$, $\Theta,\Phi \in \Upsilon_m$ *and* $0 < \mu \le 1 \le \zeta < \infty, \mu \ne \zeta$, *we have*

$$V(\Theta,\Phi) \le \left[h(\Theta,\Phi) + B(\Theta,\Phi) - \sum_{p=1}^{m} \phi_p^2\right]\left[\sum_{l=1}^{m-1} \left|\frac{\theta_{l+1}}{\phi_{l+1}} - \frac{\theta_l}{\phi_l}\right|\right], \tag{25}$$

where

$$\Xi_g(\Theta,\Phi) = \frac{1}{2}\sum_{p=1}^{m} \left|\theta_p - \phi_p\right| = \frac{1}{2}V(\Theta,\Phi). \tag{26}$$

$V(\Theta,\Phi)$ is the Variational distance [33].

Result 3.7 *For* $g(y) = \frac{(y-1)^2}{\sqrt{y}}$, $\Theta,\Phi \in \Upsilon_m$ *and* $0 < \mu \le 1 \le \zeta < \infty, \mu \ne \zeta$, *we have*

a. If $0 < \mu < 1$, then

$$L^*(\Theta,\Phi) \le \frac{1}{2}\left[h(\Theta,\Phi) + B(\Theta,\Phi) - \sum_{p=1}^{m} \phi_p^2\right]\left[\sum_{l=1}^{m-1} \left|\frac{\theta_{l+1}}{\phi_{l+1}} - \frac{\theta_l}{\phi_l}\right|\right]\left[\frac{(1-\mu)(1+3\mu)}{\mu^{\frac{3}{2}}}\right]. \tag{27}$$

b. If $\mu = 1$, then

$$L^*(\Theta,\Phi) \le \frac{1}{2}\left[h(\Theta,\Phi) + B(\Theta,\Phi) - \sum_{p=1}^{m} \phi_p^2\right]\left[\sum_{l=1}^{m-1} \left|\frac{\theta_{l+1}}{\phi_{l+1}} - \frac{\theta_l}{\phi_l}\right|\right]\left[\frac{(\zeta-1)(1+3\zeta)}{\zeta^{\frac{3}{2}}}\right], \tag{28}$$

where

$$\Xi_g(\Theta,\Phi) = \frac{1}{2}\sum_{p=1}^{m} \frac{\left(\theta_p-\phi_p\right)^2}{\sqrt{2q_p(\theta_p+\phi_p)}} = L^*(\Theta,\Phi). \tag{29}$$

$L^*(\Theta,\Phi)$ is a discrimination measure taken from [24].

Result 3.8 *For* $g(y) = \left(\frac{y+1}{2}\right) log\left(\frac{y+1}{2y}\right)$, $\Theta,\Phi \in \Upsilon_m$ *and* $0 < \mu \le 1 \le \zeta < \infty, \mu \ne \zeta$, *we have*

a. If $0 < \mu < 1$, then

$$M^*(\Theta,\Phi) \le \frac{1}{2}\left[h(\Theta,\Phi) + B(\Theta,\Phi) - \sum_{p=1}^{m} \phi_p^2\right]\left[\sum_{l=1}^{m-1} \left|\frac{\theta_{l+1}}{\phi_{l+1}} - \frac{\theta_l}{\phi_l}\right|\right]\left[\frac{1}{\mu} + \log\frac{2\mu}{\mu+1}\right]. \tag{30}$$

b. If $\mu = 1$, then

$$M^*(\Theta,\Phi) \le \frac{1}{2}\left[h(\Theta,\Phi) + B(\Theta,\Phi) - \sum_{p=1}^{m} \phi_p^2\right]\left[\sum_{l=1}^{m-1} \left|\frac{\theta_{l+1}}{\phi_{l+1}} - \frac{\theta_l}{\phi_l}\right|\right]\left[\log\frac{\zeta+1}{2\zeta} - \frac{1}{\zeta}\right], \tag{31}$$

where

$$\Xi_g(\Theta,\Phi) = \sum_{p=1}^{m} \left(\frac{\theta_p+3\phi_p}{4}\right)\log\left[\frac{\theta_p+3\phi_p}{2(\theta_p+\phi_p)}\right] = M^*(\Theta,\Phi). \tag{32}$$

$M^*(\Theta,\Phi)$ is a discrimination measure taken from [21].

Remark 3.1 Following inequality can be cited from the article [25].

$$N_1^*(\Theta,\Phi) - N_2^*(\Theta,\Phi) \le \Delta(\Theta,\Phi). \tag{33}$$

Following inequality can be cited from the article [26].

$$2\Delta(\Theta,\Phi)-\frac{1}{2}\psi(\Theta,\Phi)\leq\chi^2(\Theta,\Phi). \tag{34}$$

Following two inequalities can be seen from the article [24], for $0<\mu\leq 1\leq\zeta<\infty, \mu\neq\zeta$.

$$\frac{1}{16}\left(6\mu^{\frac{1}{2}}+\mu^{\frac{-1}{2}}+4\mu^{\frac{-3}{2}}-3\mu^{\frac{-5}{2}}\right)K(\Theta,\Phi)\leq L^*(\Theta,\Phi). \tag{35}$$

$$\frac{3\zeta^2+2\zeta+3}{32\zeta^{\frac{5}{2}}}\chi^2(\Theta,\Phi)\leq L^*(\Theta,\Phi). \tag{36}$$

Similarly following inequality can be seen from the article [21].

$$\frac{1}{4}[g(\Phi,\Theta)-G(\Phi,\Theta)]\leq M^*(\Theta,\Phi). \tag{37}$$

Now we can have some new relations among discriminations. These are as follows:

By taking (8), (9) and (33) together, we have

a. If $0<\mu<1$, then

$$N_1^*(\Theta,\Phi)-N_2^*(\Theta,\Phi)\leq\Delta(\Theta,\Phi)$$

$$\leq\left[h(\Theta,\Phi)+B(\Theta,\Phi)-\sum_{p=1}^{m}\phi_p^2\right]\left[\sum_{l=1}^{m-1}\left|\frac{\theta_{l+1}}{\phi_{l+1}}-\frac{\theta_l}{\phi_l}\right|\right]\left[\frac{\zeta^2-\mu^2}{\mu^2\zeta^2}+\left|\frac{\zeta^2+\mu^2}{\mu^2\zeta^2}-2\right|\right]. \tag{38}$$

b. If $\mu=1$, then

$$N_1^*(\Theta,\Phi)-N_2^*(\Theta,\Phi)\leq\Delta(\Theta,\Phi)\leq 2\left[h(\Theta,\Phi)+B(\Theta,\Phi)-\sum_{p=1}^{m}\phi_p^2\right]\left[\sum_{l=1}^{m-1}\left|\frac{\theta_{l+1}}{\phi_{l+1}}-\frac{\theta_l}{\phi_l}\right|\right]\left(\frac{\zeta^2-1}{\zeta^2}\right). \tag{39}$$

By taking (22), (23) and (34) together, we have

a. If $0<\mu<1$, then

b.

$$2\Delta(\Theta,\Phi)-\frac{1}{2}\psi(\Theta,\Phi)\leq\chi^2(\Theta,\Phi)$$

$$\leq 4\left[h(\Theta,\Phi)+B(\Theta,\Phi)-\sum_{p=1}^{m}\phi_p^2\right]\left[\sum_{l=1}^{m-1}\left|\frac{\theta_{l+1}}{\phi_{l+1}}-\frac{\theta_l}{\phi_l}\right|\right][(\zeta-\mu)+|2-(\zeta+\mu)|]. \tag{40}$$

b. If $\mu=1$, then

$$2\Delta(\Theta,\Phi)-\frac{1}{2}\psi(\Theta,\Phi)\leq\chi^2(\Theta,\Phi)\leq 8\left[h(\Theta,\Phi)+B(\Theta,\Phi)-\sum_{p=1}^{m}\phi_p^2\right]\left[\sum_{l=1}^{m-1}\left|\frac{\theta_{l+1}}{\phi_{l+1}}-\frac{\theta_l}{\phi_l}\right|\right](\zeta-1). \tag{41}$$

By taking (27), (28) and (35) together, we have

a. If $0<\mu<1$, then

$$\frac{1}{16}\left(6\mu^{\frac{1}{2}}+\mu^{\frac{-1}{2}}+4\mu^{\frac{-3}{2}}-3\mu^{\frac{-5}{2}}\right)K(\Theta,\Phi)\leq L^*(\Theta,\Phi)$$

$$\leq\frac{1}{2}\left[h(\Theta,\Phi)+B(\Theta,\Phi)-\sum_{p=1}^{m}\phi_p^2\right]\left[\sum_{l=1}^{m-1}\left|\frac{\theta_{l+1}}{\phi_{l+1}}-\frac{\theta_l}{\phi_l}\right|\right]\left[\frac{(1-\mu)(1+3\mu)}{\mu^{\frac{3}{2}}}\right]. \tag{42}$$

b. If $\mu=1$, then

$$\frac{1}{16}\left(6\mu^{\frac{1}{2}}+\mu^{\frac{-1}{2}}+4\mu^{\frac{-3}{2}}-3\mu^{\frac{-5}{2}}\right)K(\Theta,\Phi)\leq L^*(\Theta,\Phi)$$

$$\leq\frac{1}{2}\left[h(\Theta,\Phi)+B(\Theta,\Phi)-\sum_{p=1}^{m}\phi_p^2\right]\left[\sum_{l=1}^{m-1}\left|\frac{\theta_{l+1}}{\phi_{l+1}}-\frac{\theta_l}{\phi_l}\right|\right]\left[\frac{(\zeta-1)(1+3\zeta)}{\zeta^{\frac{3}{2}}}\right]. \tag{43}$$

By taking (27), (28) and (36) together, we have

a. If $0 < \mu < 1$, then

$$\frac{3\zeta^2+2\zeta+3}{32\zeta^{\frac{5}{2}}}\chi^2(\Theta,\Phi) \leq L^*(\Theta,\Phi)$$

$$\leq \frac{1}{2}\left[h(\Theta,\Phi)+B(\Theta,\Phi)-\sum_{p=1}^{m}\phi_p^2\right]\left[\sum_{l=1}^{m-1}\left|\frac{\theta_{l+1}}{\phi_{l+1}}-\frac{\theta_l}{\phi_l}\right|\right]\left[\frac{(1-\mu)(1+3\mu)}{\mu^{\frac{3}{2}}}\right]. \quad (44)$$

b. If $\mu = 1$, then

$$\frac{3\zeta^2+2\zeta+3}{32\zeta^{\frac{5}{2}}}\chi^2(\Theta,\Phi) \leq L^*(\Theta,\Phi)$$

$$\leq \frac{1}{2}\left[h(\Theta,\Phi)+B(\Theta,\Phi)-\sum_{p=1}^{m}\phi_p^2\right]\left[\sum_{l=1}^{m-1}\left|\frac{\theta_{l+1}}{\phi_{l+1}}-\frac{\theta_l}{\phi_l}\right|\right]\left[\frac{(\zeta-1)(1+3\zeta)}{\zeta^{\frac{3}{2}}}\right]. \quad (45)$$

By taking (30), (31) and (37) together, we have

a. If $0 < \mu < 1$, then

$$\frac{1}{4}[g(\Phi,\Theta)-G(\Phi,\Theta)] \leq M^*(\Theta,\Phi)$$

$$\leq \frac{1}{2}\left[h(\Theta,\Phi)+B(\Theta,\Phi)-\sum_{p=1}^{m}\phi_p^2\right]\left[\sum_{l=1}^{m-1}\left|\frac{\theta_{l+1}}{\phi_{l+1}}-\frac{\theta_l}{\phi_l}\right|\right]\left[\frac{1}{\mu}+\log\frac{2\mu}{\mu+1}\right]. \quad (46)$$

b. If $\mu = 1$, then

$$\frac{1}{4}[g(\Phi,\Theta)-G(\Phi,\Theta)] \leq M^*(\Theta,\Phi)$$

$$\leq \frac{1}{2}\left[h(\Theta,\Phi)+B(\Theta,\Phi)-\sum_{p=1}^{m}\phi_p^2\right]\left[\sum_{l=1}^{m-1}\left|\frac{\theta_{l+1}}{\phi_{l+1}}-\frac{\theta_l}{\phi_l}\right|\right]\left[\log\frac{\zeta+1}{2\zeta}-\frac{1}{\zeta}\right]. \quad (47)$$

Where

$$N_k^*(\Theta,\Phi) = \sum_{p=1}^{m}\frac{(\theta_p-\phi_p)^{2k}}{(\theta_p+\phi_p)^{2k-1}}\exp\frac{(\theta_p-\phi_p)^2}{(\theta_p+\phi_p)^2}, k = 1,2,3,\ldots$$

are Jain and Saraswat discriminations [28],

$$\psi(\Theta,\Phi) = \sum_{p=1}^{m}\frac{(\theta_p-\phi_p)^2(\theta_p+\phi_p)}{\theta_p\phi_p}$$

is the Symmetric Chi- square Discrimination [13] and

$$K(\Theta,\Phi) = \sum_{p=1}^{m}\theta_p\log\frac{\theta_p}{\phi_p}$$

is the Relative information or Kullback- Leibler discrimination or Directed discrimination or Information gain [34].

4. Mathematical Validation of the Obtained Results

Presently, we'll confirm scientifically a few gotten results and relations, like: (8), (19), (22), (25), (30) and (40).

For this, let Θ be the binomial probability distribution (Real data) with parameters $(m = 10, \theta = 0.7)$ (m is the total finite trials and θ is the probability of the success of one trial) and Φ be a Poisson probability distribution (approximated data) with parameter $(\xi = m\theta = 7)$ for the random variable , then we have

Table 1 - $(\boldsymbol{m = 10, \theta = 0.7, \phi = 0.3})$

x_p	0	1	2	3	4	5	6	7	8	9	10
$\theta_p \approx$	.0000059	.000137	.00144	.009	.036	.102	.200	.266	.233	.121	.0282
$\phi_p \approx$	.000911	.00638	.022	.052	.091	.177	.199	.149	.130	.101	.0709
$\frac{\theta_p+\phi_p}{2\phi_p} \approx$	.503	.510	.532	.586	.697	.788	1.002	1.392	1.396	1.099	.698

By using Table, we have

$$\mu(= .503) \leq \frac{\theta_p+\phi_p}{2\phi_p} \leq \zeta(= 1.396). \tag{48}$$

$$\Delta(\Theta,\Phi) = \sum_{p=1}^{11} \frac{(\theta_p-\phi_p)^2}{\theta_p+\phi_p} \approx .1812. \tag{49}$$

$$\chi^2(\Theta,\Phi) = \sum_{p=1}^{11} \frac{\left(\theta_p-\phi_p\right)^2}{\phi_p} \approx .3298. \tag{50}$$

$$V(\Theta,\Phi) = \sum_{p=1}^{11} \left|\theta_p - \phi_p\right| \approx .4844. \tag{51}$$

$$h(\Theta,\Phi) = \sum_{p=1}^{11} \frac{\left(\sqrt{\theta_p}-\sqrt{\phi_p}\right)^2}{2} \approx .0502155. \tag{52}$$

$$B(\Theta,\Phi) = \sum_{p=1}^{11} \sqrt{\theta_p\phi_p} \approx .9978542. \tag{53}$$

$$\psi(\Theta,\Phi) = \sum_{p=1}^{11} \frac{\left(\theta_p-\phi_p\right)^2(\theta_p+\phi_p)}{\theta_p\phi_p} \approx 1.5558. \tag{54}$$

$$J_R(\Theta,\Phi) = \sum_{p=1}^{11} \left(\theta_p - \phi_p\right)\log\left(\frac{\theta_p+\phi_p}{2\phi_p}\right) \approx .0808. \tag{55}$$

$$M^*(\Theta,\Phi) = \sum_{p=1}^{11} \left(\frac{\theta_p+3\phi_p}{4}\right)\log\left[\frac{\theta_p+3\phi_p}{2(\theta_p+\phi_p)}\right] \approx .0076525. \tag{56}$$

Now put the approximated numerical values from equations (48) to (56) into results (8), (19), (22), (25), (30) and (40). We have

$$\Rightarrow 0.1812\big(= \Delta(\Theta,\Phi)\big) \leq \big[0.0502155\big(= h(\Theta,\Phi)\big) + 0.9978542\big(= B(\Theta,\Phi)\big) - 0.13676\big(= \textstyle\sum_{p=1}^{11} \phi_p^2\big)\big]$$

$$\times \left[3.180212\left(= \sum_{l=1}^{10} \left|\frac{\theta_{l+1}}{\phi_{l+1}} - \frac{\theta_l}{\phi_l}\right|\right)\right] \times 5.90485.$$

$$\Rightarrow 0.1812 \leq 2.89815 \times 5.90485.$$

$$\Rightarrow 0.1812 \leq 17.113188,$$

hence validated the result (8).

$$0.0808\big(= J_R(\Theta,\Phi)\big) \leq 2 \times 2.89815 \times 1.67522 \Rightarrow 0.0808 \leq 9.710077,$$

hence validated the result (19).

$$0.3298\big(= \chi^2(\Theta,\Phi)\big) \leq 4 \times 2.89815 \times 0.994 \Rightarrow 0.3298 \leq 11.523,$$

hence validated the result (22).

$$0.4844\big(= V(\Theta, \Phi)\big) \le 2.89815,$$

hence validated the result (25).

$$0.0076525\big(= M^*(\Theta, \Phi)\big) \le \frac{1}{2} \times 2.89815 \times 1.5865 \Rightarrow 0.0076525 \le 2.299088,$$

hence validated the result (30).

$$\left(2 \times 0.1812 - \frac{1}{2} \times 1.5558\right)\left(= 2\Delta(\Theta, \Phi) - \frac{1}{2}\psi(\Theta, \Phi)\right) \le 0.3298\big(= \chi^2(\Theta, \Phi)\big) \le 4 \times 2.89815 \times 0.994$$

$$\Rightarrow -0.4155 \le 0.3298 \le 11.5230,$$

hence validated the result (40).

So also, other results can be confirmed. Too, approval of all the over results can be done by utilizing distinctive values of m and θ and for other discrete likelihood dispersions as well, like: Negative binomial, Geometric, uniform etc.

5. Conclusion

Since discrimination measures have wide applications in a few areas, so it is continuously curiously and critical to discover modern disparities and results in numerical shapes as well, so that these can be connected as an applications in numeric shapes. Since results are very unique, way better and compact to the past discoveries, subsequently these can moreover be connected within the disciplines said in the introduction section. Motivation of this work is to discover the unused realtions among the well known separation measures with Bhattachrya and Hellinger discriminations by employing a determined disparity on Jain- saraswat's functional discrimination degree. The results are unique to the leading of author's information.

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