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Centroidal Polygon: A New Enhancement of Euler to Improve Accuracy of First Order Non-Linear Ordinary Differential Equation

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Abstract: The Euler method is one of the oldest methods to solve differential equation problems. The Euler method produces the simplest solution. However, although is not computationally expensive, the Euler method is lack of accuracy. To improve the Euler method, the researcher proposed a new scheme for better accuracy. The Euler method equation and the mean method were combined to enhance this method. As the improvement basis, the researcher used the Centroidal mean and the midpoint method or Polygon to improve the Euler method. The combination of the Euler and Centroidal mean is known as Centroidal Polygon (CP) scheme. The CP scheme was used to solve first-order non-linear Ordinary Differential Equations (ODE). The researcher used SCILAB 6.0 software to solve the equation and the CP scheme was tested in three different step sizes (0.1,0.01, and 0.001). Aside from that, the researcher had compared the CP scheme with previous schemes such as ZulZamri's Polygon (P) scheme, Nurhafizah's Harmoni-Polygon (HP) scheme, and Nooraida's Cube-Polygon (CuP) scheme to ensure that the CP scheme is more accurate than previous research. When the maximum error is calculated by subtracting the scheme and exact solution, the results show that the CP scheme delivers the highest accuracy results in the shortest amount of time. The new enhanced, modified Euler method is useful for other researchers to achieve good accuracy at low computational cost as an alternative to the more computationally expensive methods.

Keywords: First order non-linear ODE, centroidal mean, accuracy, Euler method

1. Introduction

The numerical method is a technique for resolving mathematical and computational problems by performing arithmetic operations. The equation can be solved using a number of different numerical methods. The oldest method for solving an arithmetic problem is the Euler method. The Euler method was one of the first and most common enhancement methods before Runge Kutta and Adam Bashforth. All one-step methods can be interpreted in a general form, except how the slope is measured. As with the falling parachutist problem, the most straightforward approach is to use the differential equation to estimate the first derivative type slope at the beginning of the interval, with the slope taken as an approximation of the average slope over the entire interval[1].

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In the numerical solution of the ordinary differential equation, there are two types of errors. The first error is truncation, followed by rounding off. The techniques used to estimate the value of y results in truncation, which is incorrect. Therefore, the researcher uses the SCILAB 6.0 software to examine the problem.

The second error is rounding off. Rounding off is caused by a computer's failure to retain a certain number of significant digits. When multiplying or dividing measured values, the final answer can contain only as many significant figures as the least precise value. The final answer cannot contain more decimal places than the least precise value[2]. The researcher decided to set the solution into six decimal places to ensure the solution from all schemes gets an accurate value. As a result, the decimal place is less of a miscalculation, and it is more important to remember that the method used is the most accurate.

The Euler method was previously only used in mathematical fields. The researcher would apply the test solution to other fields to see whether the Euler principle could be applied to them, especially in engineering. The Euler method has also been demonstrated in Mechanical Engineering applications, including fluid dynamic computations for flow analysis[3], [4].

Before the coding of the non-linear equation into the SCILAB 6.0 application is performed, the non-linear equation must be solved in MS Excel. Before moving on to the non-linear equation, the researcher decided to test the new scheme on a linear equation. This is to ensure that the new scheme is suitable for implementation. To enhance the Euler method, there are two methods that were previously used in other researches. They are Heun's method and the midpoint method[5]. The Heun method, which involves deciding two derivatives for the interval, improves slope estimation. One at the beginning and one at the end of the point. The slope estimate for the entire interval is then boosted using the two derivatives. A second improved technique for improving the Euler method is the midpoint method, also known as Improved Polygon[4]. This method is used to calculate the value of y at the midpoint of the interval. Based on the previous study, the researcher decided to reinforce the Euler equation using the modern Centroidal Polygon (CP) scheme and a midpoint method as the base method[6].

Previous studies on polygons and harmonic polygons will be compared to this study. The researcher can compare the precision of the solution to previous studies as well as the computational speed. Previous research shows that centroidal mean will provide a more accurate solution than the other mean such as arithmetic mean and geometric mean[7] . To enhance the Euler method, the researcher will use the Centroidal Polygon (CP) scheme and conducting evaluation by using the first order non-linear ODE. The first order non-linear ODE will be compared to ZulZamri's polygon and Nurhafizah's Harmonic-polygon and Cube-Polygon Nooraida[8].

$$y_{n+1} = y_n + hf(x_n, y_{n)} \quad [1] \tag{1}$$

The Euler method and the combination of Euler's method and the mean principle with a distinct mean form will be included in the equation. As suggested in this research, the centroidal mean is compared to other previous approaches that often use the Euler and mean concept combinations. Differential step sizes of 0.1, 0.01, and 0.001 were used in the schemes[9]. The study's main aim is to improve the Euler method using the Centroidal Polygon scheme. This research aimed to see if the Centroidal Polygon scheme solutions could be applied to first order non-linear ODE. Apart from that, the investigation aimed to see how effective the Centroidal Polygon scheme was at solving first-order non-linear ODE.

2. Methodology

There are two main methods used to improved Euler, which are Heun method and midpoint method. This research used midpoint method and applied into the new scheme. This method is called as improved Polygon method[1]. In Polygon method, it utilizes a slope estimate at the midpoint of the prediction interval. Thus, this midpoint method is being applied into the new scheme. Figure 1 shows new modified Euler (CP) scheme is developed. This scheme is developed by combining Euler (E) and means (M). The original Euler scheme with the general formula n + 1 = y_n + hf (x_n, y_n) is selected as the basis for developing a scheme. Centroidal mean selected in developing this proposed scheme which is centroidal mean (M). This combination of Euler (E) and mean (M) produces proposed scheme known as Centroidal Polygon (CP) (E+M)[10], [11].

Euler		Mean		Proposed Scheme
• E		• M		• E+M = CP
yn+1=yn+hf(xn,yn) *n=1,2,3,....*	+	$M = \frac{2((x_n)^2 + x_n y_n + (y_n)^2)}{3(x_n + y_n)}$	=	CP= $y_n + hf\left(\frac{x_n+(x_n+h)}{2}, \frac{y_n+(y_n+f(x_n,y_n))}{2}\right) + \frac{2((x_n)^2+x_n y_n+(y_n)^2)}{3(x_n+y_n)}$ $y_{n+1} = y_n + hf\left(\frac{2((x_n)^2+x_n(x_n+h)+(x_n)^2}{3(x_n+x_n+h)}, \frac{2((y_n)^2+y_n(y_n+hf(x_n,y_n))+(y_n+hf(x_n,y_n))^2}{3(y_n+y_n+hf(x_n,y_n))}\right)$

Fig. 1 - The Centroidal-Polygon scheme form

This analysis will use four different schemes: The Polygon scheme, Harmonic-Polygon scheme, Cube-Polygon scheme, and Centroidal Polygon scheme. The researcher will evaluate the linear equation to ensure that the equation is computable before continuing the non-linear equation[11]. The researcher will construct the mathematical program method as follows before beginning to test the schemes into the non-linear first order ODE:

a) Evaluate the Centroidal structure with previous schemes

b) Algorithm creation

c) Details of documentation

The Centroidal Polygon scheme will be compared to previous schemes using the SCILAB 6.0 program. The researcher will manually compute the MS Excel solutions before using SCILAB 6.0 software to ensure that the Centroidal Polygon scheme is computable in SCILAB 6.0 software[12]. By tabulating the maximum error, all schemes will be compared with the exact solution.

Table 1 shows the non-linear first order ODE structure. All of the values and equations are derived from the sources[2], [4], [13] to ensure that the value can be used. Three different types of equations will be put to the test.

The centroidal mean equation will be added to the function f(x,y) in the Euler equation for the Centroidal Polygon scheme. The researcher will explain how the Centroidal Polygon scheme implied in the equation is implemented (2). Based on equation (3), the researcher would boost the Euler equation (4).

$$y_{n+1} = y_n + \Delta t\, f(x_n, y_n) \tag{2}$$

The improved Euler are using the mean concept. The mean concept in this researcher uses Centroidal mean, which may be considered a centre of two points as written in equations (3) and (4).

$$\frac{2x_0^2+x_0x_1+x_1^2}{3x_0+x_1}, \frac{2y_0^2+y_0y_1+y_1^2}{3y_0+y_1} \tag{3}$$

$$\frac{2x_0^2+x_0(x_0+h)+(x_0+h)^2}{3x_0+(x_0+h)}, \frac{2y_0^2+y_0(y_0+h)+(y_0+h)^2}{3y_0+(y_0+h)} \tag{4}$$

The Centroidal equation in equation (4) will be implemented into the equation (3) to improve the equation. The new equation formed are as in equation (5)

$$\frac{y-y_0}{h} = f\left(\frac{2x_0^2+x_0(x_0+h)+(x_0+h)^2}{3x_0+(x_0+h)}, \frac{2y_0^2+y_0(y_0+h)+(y_0+h)^2}{3y_0+(y_0+h)}\right)$$

$$y - y_0 = hf\left(\frac{2x_0^2+x_0(x_0+h)+(x_0+h)^2}{3x_0+(x_0+h)}, \frac{2y_0^2+y_0(y_0+h)+(y_0+h)^2}{3y_0+(y_0+h)}\right)$$

$$y = y_0 + hf\left(\frac{2x_0^2 + x_0(x_0+h) + (x_0+h)^2}{3x_0 + (x_0+h)}, \frac{2y_0^2 + y_0(y_0+h) + (y_0+h)^2}{3y_0 + (y_0+h)}\right) \quad (5)$$

Euler can be more accurate at a lower computational cost by modifying the equation using the combination of Euler method and mean concept (5**)**

3. Result and Discussion

This chapter shows the result of the comparison of the three modified Euler schemes with the exact solution. Table 1 shows the first order non-linear ODE problems to be used and Table 2 shows the maximum error between the Centroidal Polygon scheme with the Polygon and Harmonic Polygon schemes. To ensure the CP scheme can solve and get better accuracy, the researcher tests three different non-linear equations. Other than that, all schemes will be tested into three different step sizes, just as explained in the introduction.

The result of the maximum error comes from subtracting the scheme's solution from the exact solution. Thus, whose scheme gets the maximum error near to zero, so that scheme is more accurate. The solution in Table 2 is determined as;

Error = $[E_x - E_v]$;

E_x = Exact solution and E_v = Euler's improved value

Table 1 - Set of problem first order non-linear ODE

Equation	Exact Solution	Initial Value	Interval of Integration	Sources
$y' = y(1-y)^3$	$y(x) = \frac{0.5}{(1-0.5)e^{-x+0.5}}$	$y(0) = 0.5$	$0 \le x \le 1$	**[10]**
$y' = -(1-y)^2$	$y(x) = 1 + \frac{1}{x+1}$	$y(0) = 2$	$0 \le x \le 20$	**[15]**
$y' = -y^4$	$y(x) = \frac{1}{\sqrt[3]{x+1}}$	$y(0) = 1$	$0 \le x \le 1$	**[13]**

Table 2 - Result for maximum error of first order non-linear Ordinary Differential Equation (ODE) problem

Scheme	**Polygon**			**Harmonic-Polygon**			**Cube-Polygon**			**Centroidal Polygon**		
Step size	**0.1**	**0.01**	**0.001**	**0.1**	**0.01**	**0.001**	**0.1**	**0.01**	**0.001**	**0.1**	**0.01**	**0.001**
Problem 1	0.150226	0.149384	0.149263	0.178763	0.177558	0.177399	0.167196	0.164275	0.163968	0.147861	0.146055	0.145848
Problem 2	0.119307	0.093233	0.090882	0.225009	0.184378	0.181007	0.130673	0.101228	0.098650	0.112425	0.087688	0.085956
Problem 3	0.581656	0.658021	0.665801	0.531656	0.653021	0.665301	0.552286	0.655084	0.665507	0.498323	0.649688	0.664967

The result of the first order non-linear ODE problem is shown in Table 2. The Centroidal Polygon (CP) scheme matched the troubleshooting results for Problem 1, Problem 2 and Problem 3 after observation because it gave an accurate value compared to the other three schemes. All schemes were tested into three different step sizes. There are 0.1, 0.01 and 0.001.

For Problem 1, the CP scheme yielded a maximum error of 0.147861 at step size 0.1, while the P scheme yielded a maximum error of 0.150226, the HP scheme yielded a maximum error of 0.178763, and the CuP scheme yielded a maximum error of 0.167196. The CP scheme had a maximum error of 0.146055 at step size 0.01, while the P, HP, and CuP schemes had maximum errors of 0.149384, 0.177558, and 0.164275, respectively. Finally, with a step size of 0.001, the CP scheme had a maximum error of 0.145848, while the other three schemes (P, HP, and CuP) had maximum errors of 0.149263, 0.177399, and 0.163968, respectively.

The maximum error for the CP scheme, P scheme, HP scheme, and CuP scheme is shown in the graph below. It is can be shown that CP scheme presented in the red line is consistently most accurate at a higher step size compared to the other schemes.

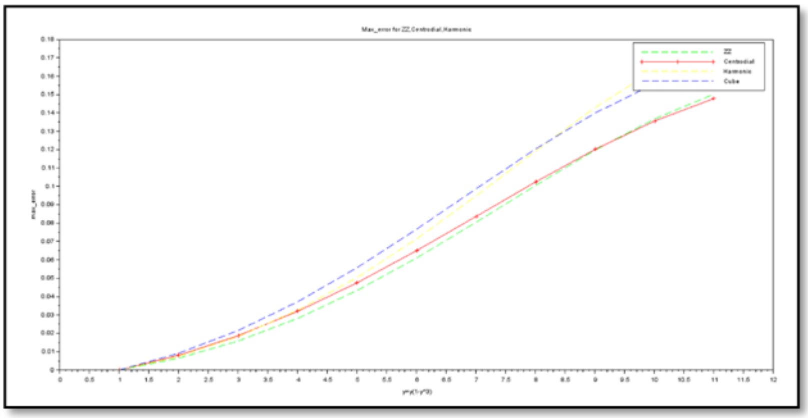

Fig. 2 - Maximum error vs step size 0.1 for Problem 1

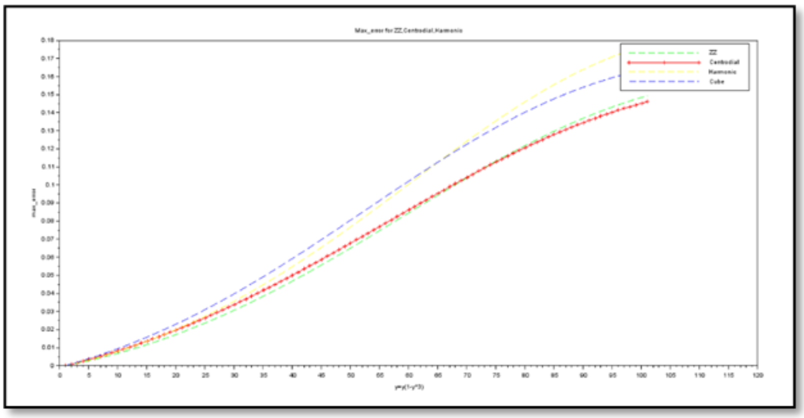

Fig. 3 - Maximum error vs step size 0.01 for Problem 1

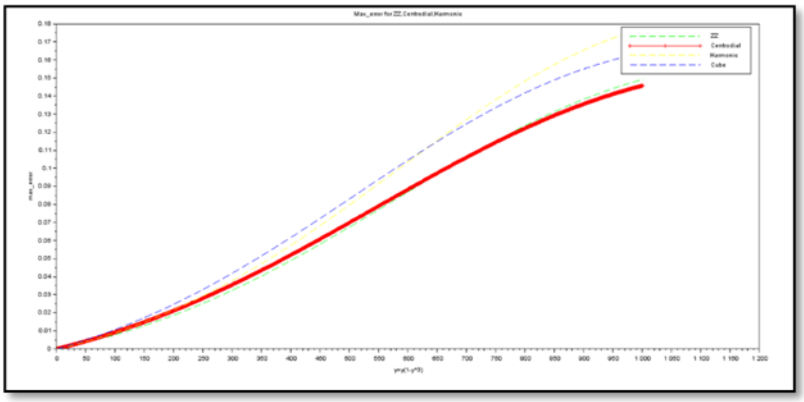

Fig. 4 - Maximum error vs step size 0.001 for Problem 1

For Problem 2, the CP scheme had a maximum error of 0.112425 at step size 0.1, while the P scheme had a maximum error of 0.119307, the HP scheme had a maximum error of 0.225009, and the CuP scheme had a maximum error of 0.130673 at step size 0.1. The CP scheme had a maximum error of 0.087688 at step size 0.01, while the P, HP, and CuP schemes had maximum errors of 0.093233, 0.184378, and 0.101228, respectively. Finally, with a step size of 0.001, the CP scheme had a maximum error of 0.085956, while the other three schemes (P, HP, and CuP) had maximum errors of 0.090882, 0.181007, and 0.098650, respectively. The maximum error between the CP scheme, P scheme, HP scheme, and CuP scheme is shown in the graph below. It is can be shown that CP scheme presented in the red line is consistently most accurate at a higher step size compared to the other schemes.

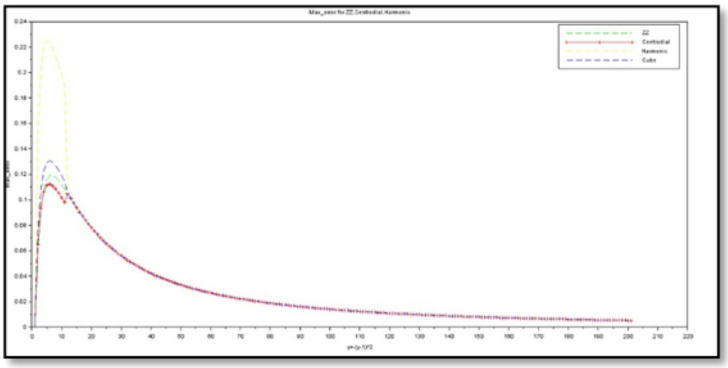

Fig. 5 - Maximum error vs step size 0.1 for Problem 2

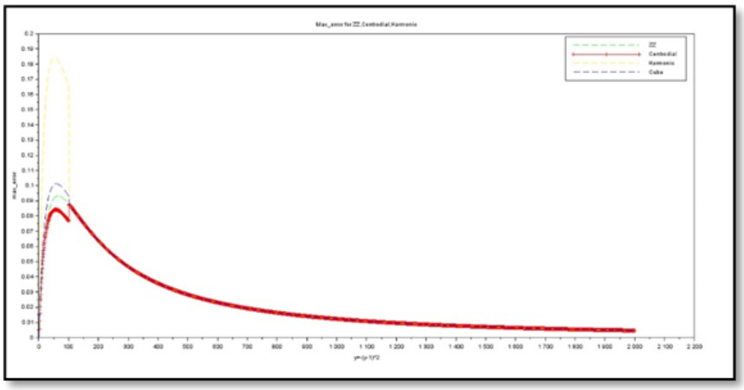

Fig. 6 - Maximum error vs step size 0.01 for Problem 2

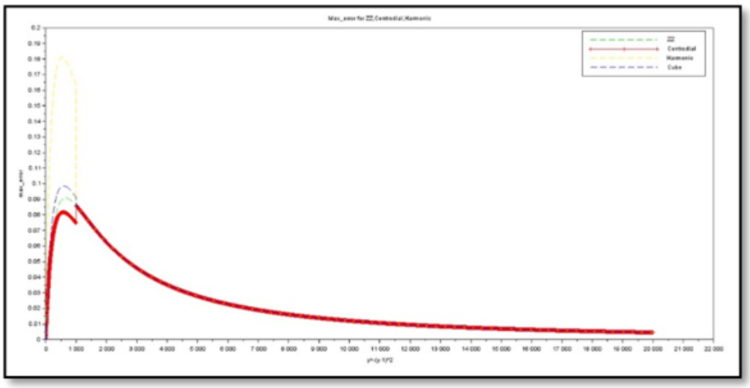

Fig. 7 - Maximum error vs step size 0.001 for Problem 2

For the Problem 3, at step size 0.1, the maximum error for the CP scheme was 0.664967, while the maximum error for the P scheme was 0.665801, the maximum error for the HP scheme was 0.665301, and the maximum error for the CuP scheme was 0.665507. The CP scheme had a maximum error of 0.649688 at step size 0.01, while the P, HP, and CuP schemes had maximum errors of 0.658021, 0.653021, and 0.655084, respectively. Finally, with a step size of 0.001, the CP scheme had a maximum error of 0.498323, while the other three schemes (P, HP, and CuP) had maximum errors of 0.581656, 0.531656, and 0.552286, respectively. The maximum error between the CP scheme, P scheme, HP scheme, and CuP scheme is shown in the graph below. It is can be shown that CP scheme presented in the red line is consistently most accurate at a higher step size compared to the other schemes.

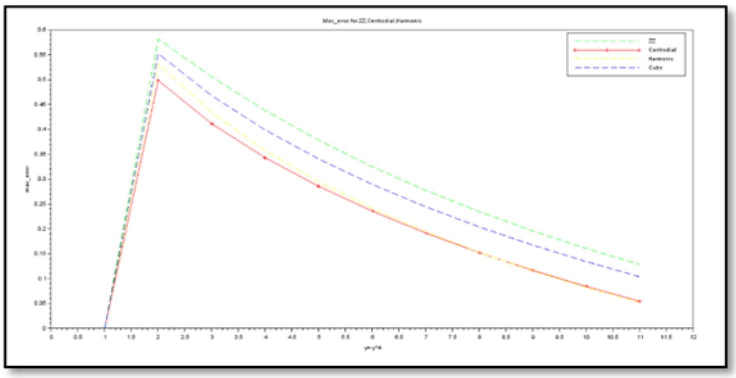

Fig. 8 - Maximum error vs step size 0.1 for Problem 3

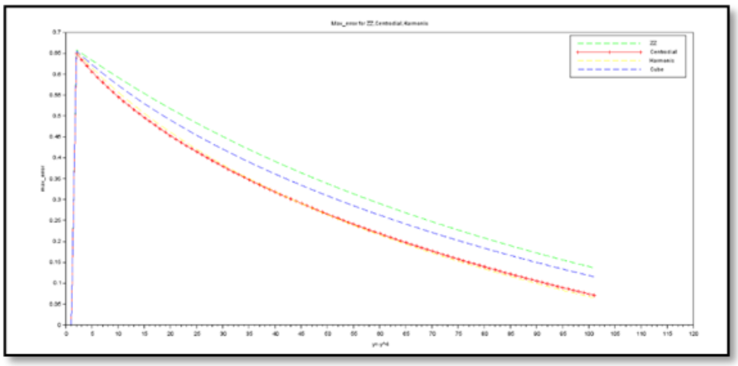

Fig. 9 - Maximum error vs step size 0.01 for Problem 3

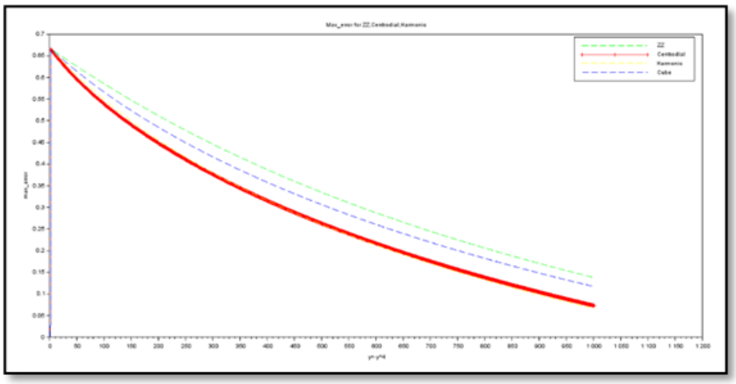

Fig. 10 - Maximum error vs step size 0.001 for Problem 3

4. Conclusion

The present analysis's key emphasis is to emphasize the maximum error caused by the first order of non-linear ODE. The most important factor for accuracy is focused on the Euler method. The CP scheme is more effective in this research than the P, HP and CuP schemes. It should be mentioned that the CP scheme can be provided more accurate compare to P, HP and CuP schemes especially at a higher step size and it is concluded the maximum error is inversely proportional to the step size. From the result, it also concluded that the CP scheme can be used as a better alternative scheme to enhance the Euler method.

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