© Universiti Tun Hussein Onn Malaysia Publisher's Office

IJIE

http://penerbit.uthm.edu.my/ojs/index.php/ijie
ISSN : 2229-838X e-ISSN : 2600-7916

The International Journal of Integrated Engineering

An Expected Average Run Length (EARL) Performance Comparison of the SSGR and EWMA Control Charts

H.W. You[1*], Z.L. Chong[2], W.L. Teoh[3], M.B.C. Khoo[4], W.C. Yeong[5]

[1]Pusat GENIUS@Pintar Negara,
Universiti Kebangsaan Malaysia, 43600 UKM Bangi, Selangor, MALAYSIA

[2]Department of Mathematical and Actuarial Sciences, Lee Kong Chian Faculty of Engineering and Science,
Universiti Tunku Abdul Rahman, Jalan Sungai Long, Bandar Sungai Long, 43000 Kajang, MALAYSIA

[3]School of Mathematical and Computer Sciences,
Heriot-Watt University Malaysia 62200 Putrajaya, MALAYSIA

[4]School of Mathematical Sciences,
Universiti Sains Malaysia, 11800 Penang, MALAYSIA

[5]School of Mathematical Sciences,
Sunway University, 47500 Petaling Jaya, MALAYSIA

*Corresponding Author

DOI: https://doi.org/10.30880/ijie.2023.15.05.005
Received 5 May 2023; Accepted 25 May 2023; Available online 19 October 2023

Abstract: The acceleration use of control charts in industrial processes has led to the effectiveness in their evaluation by quality practitioners. This is crucial, as it influences their decisions on the choice of which control charts to employ. This study aims to explore and compare the performance of the side sensitive group runs (SSGR) and exponentially weighted moving average (EWMA) control charts. In general, the average run length (ARL) characteristics were used to evaluate the performance of these control charts. The ARL, which considers the exact shift size in the process, is restricted in the case when the practitioner cannot identify the process shift size (unknown shift size). In this situation, the expected average run length (EARL) is an alternative performance criterion. Upon comparison of the findings obtained, the EWMA chart has superior performance when ($\delta_{\min}$, $\delta_{\max}$) = (0.1, 0.4). In contrast, the SSGR chart overtakes the EWMA chart when ($\delta_{\min}$, $\delta_{\max}$) = (0.5, 0.8) and ($\delta_{\min}$, $\delta_{\max}$) = (0.9, 1.2), except when the sample size n = 3 for ($\delta_{\min}$, $\delta_{\max}$) = (0.5, 0.8). For this particular combination, the EWMA chart performs slightly better than the SSGR chart. The outcome of this study is expected to contribute to practitioners in identifying suitable control charts in process monitoring and implementation.

Keywords: Expected average run length, unknown shift size, side sensitive group runs chart, exponentially weighted moving average chart

1. Introduction

Statistical Process Control (SPC) is a prominent statistical method used to monitor and enhance process quality. This is essential so that the final product is in meeting consumer satisfaction. Among the SPC tools, the control chart is the most widely employed in manufacturing and service settings [1, 2]. A control chart is a graphical approach for examining changes in a process over a specific time sequence, with the horizontal lines in the chart assisting in decision

*Corresponding author: hwyou@ukm.edu.my

making. The Shewhart chart is the first control chart introduced by Walter A. Shewhart in monitoring the process mean [3, 4]. The Shewhart chart is well known for detecting large process shifts. Accordingly, extensive research has been conducted to introduce new control charts with the aim of improving the sensitivity of the Shewhart chart [5]. To date, the exponentially weighted moving average (EWMA) chart and side sensitive group runs (SSGR) chart, to name a few, have been developed. In practice, the decision of the practitioner in selecting the use of a control chart is based on its effectiveness in detecting process shifts. In light of this, the evaluation of the performance of control charts is vital [6].

The typical performance characteristic in evaluating control charts is the average run length (ARL). The ARL is the average number of samples plotted on a control chart before an out-of-control signal is detected. The computation of ARL requires the practitioner to determine the magnitude of the process shift size [7]. With this, the shift size must be known in advance. Oftentimes, the practitioner lacks of sufficient knowledge for the next shift size in the process. In light of this, the expected ARL (EARL) can be used as an alternative performance criterion, where the EARL takes into account when the shift size is unknown [8]. Hence, the performance of the SSGR chart and EWMA chart will be compared using the EARL. These control charts are reviewed briefly in Section 2. In addition, the formula used to compute the EARL is presented in Section 2. Section 3 discusses the comparative performance of the SSGR and EWMA control charts. Finally, conclusions are drawn in Section 4.

2. Methodology

2.1 Side Sensitive Group Runs Chart

The side sensitive group runs chart is a combination of the Shewhart chart and an extended version of the conforming run length (CRL) chart [9]. It is worth noting that in the case a sample falls outside the control limit of the Shewhart chart, it indicates a nonconforming sample. An extended version of the CRL chart is needed to determine the status of the process. There is one lower limit for the CRL chart, denoted as *L*. The upper control limit (*UCL*) and lower control limit (*LCL*) of the Shewhart chart are as follows:

$$UCL/LCL = \mu_0 \pm \frac{K}{\sqrt{n}}\sigma_0, \tag{1}$$

where μ_0 and σ_0 denote the in-control mean and in-control standard deviation, respectively. Note that K is the design constant and n is the sample size. The ARL of the SSGR chart is calculated as [10]

$$\text{ARL}= \frac{1-h(1-h)A^2}{PA^2[1+h(1-h)(A-2)]}. \tag{2}$$

Let P be the probability of a nonconforming sample on the Shewhart chart, i.e.

$$\begin{aligned} P &= 1-\Pr\left(LCL \le \bar{X} \le UCL\right) \\ &= 1-\Phi\left(K-\delta\sqrt{n}\right)+\Phi\left(-K-\delta\sqrt{n}\right). \end{aligned} \tag{3}$$

Note that $\Phi(\cdot)$ denotes the standard normal cumulative distribution function (cdf). Here, A and h can be expressed as follows:

$$A = 1-(1-P)^L \tag{4}$$

And

$$h = \frac{1-\Phi\left(K-\delta\sqrt{n}\right)}{P}. \tag{5}$$

2.2 Exponentially Weighted Moving Average Chart

Assume the process is independent and identically distributed, having a normal distribution with in-control mean, μ_0 and in-control variance σ_0^2. Let Z_u be the statistic plotted on a control chart with control limits $\pm J\sqrt{\lambda/(2-\lambda)}$, where J is the width constant, i.e. $J > 0$. If Z_u falls outside the control limits, then the EWMA chart will give a signal. The Z_u is defined as follows [11]:

$$Z_u = \lambda Y_u + (1-\lambda) Z_{u-1}, \text{ for } u = 1, 2, ..., \quad (6)$$

where λ is a smoothing constant, i.e. $0 < \lambda \leq 1$ and Y_u is

$$Y_u = \frac{\bar{X}_u - \mu_0}{\sigma_0 / \sqrt{n}}. \quad (7)$$

Note that $\bar{X}_u$ represents the mean of the uth sample, and n is the sample size. The ARL of the EWMA chart will be evaluated using the Markov chain approach [12]. Let

$$\mathbf{D} = \left[D_{i,j} \right]_{g \times g} \quad (8)$$

be the transition probability matrix (tpm) for the transient states, where $i, j = 1, 2, \ldots g$. The interval between the control limits, i.e. LCL and UCL will be divided into $g = 2l+1$ subintervals and each of width $2q$, where $q = (UCL - LCL)/(2g)$. Let H_j represents the midpoint of the jth subinterval, for $j = 1, 2, \ldots, 2l+1$. Then the entries, $D_{i,j}$ of matrix $\mathbf{D}$ are

$$D_{i,j} = \Phi\left(\frac{H_j + q - (1-\lambda)H_i}{\lambda} - \delta \right) - \Phi\left(\frac{H_j - q - (1-\lambda)H_i}{\lambda} - \delta \right) \quad (9)$$

for $i, j = 1, 2, \ldots, 2l+1$, where $\delta = \left(\sqrt{n} \left| \mu - \mu_0 \right| \right) / \sigma_0$ with μ is the out-of-control mean. Finally, the ARL for the EWMA chart can be calculated using the following equation:

$$\text{ARL} = \mathbf{d}^T (\mathbf{I} - \mathbf{D})^{-1} \mathbf{1} \quad (10)$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{1} = (1, 1, \ldots, 1)^T$ and $\mathbf{d}$ is a vector of initial probabilities having entries

$$d_j = \begin{cases} 1, & \text{if } j = l+1 \\ 0, & \text{otherwise} \end{cases} \quad (11)$$

for $j = 1, 2, \ldots, 2l+1$, where $Z_0 = 0$ is the starting value of the EWMA statistics in Equation (6).

2.3 Formula for Expected Average Run Length

The assumption in evaluating the performance of the control chart is that the exact shift size is known, where it is considered a deterministic quantity. In monitoring a process, the shift size for the next shift is rarely known in practice. Hence, the shift size should be considered as a random variable. When the magnitude of the shift size is unknown, it is essential to consider the EARL for an overall range of shifts ($\delta_{\min}$, $\delta_{\max}$). Here, $\delta_{\min}$ is the lower bound of the mean shift, and $\delta_{\max}$ is the upper bound of the mean shift. The EARL is computed as:

$$\text{EARL} = \int_{\delta_{\min}}^{\delta_{\max}} f_\delta(\delta) \text{ARL } d\delta, \quad (12)$$

where $f_\delta(\delta)$ is the probability density function (pdf) of the shift size, δ. Note that the ARL in Equation (12) can be replaced with the ARL from SSGR and EWMA charts in Equations (2) and (10), respectively.

3. Results and Discussion

In the application of control charts, the magnitude of shift size is seldom known in practice. In this situation, the EARL criterion is used to measure the performance of the control chart. Note that two EARLs are usually of interest, namely, the in-control EARL, EARL_0, and the out-of-control EARL, EARL_1. Here, the performance of the SSGR chart and EWMA chart were compared based on the corresponding optimal charting parameters for the sample sizes, $n \in \{3, 5, 7, 9\}$. Three different ranges of shifts ($\delta_{\min}$, $\delta_{\max}$) are considered here, i.e., (0.1, 0.4), (0.5, 0.8) and (0.9, 1.2). The

$EARL_0$ is taken to be 370.4. Tables 1 and 2 presented the optimal charting parameters for the SSGR chart and EWMA chart, respectively. The results are demonstrated in Table 3. For example, by considering n = 5, (δ_{min}, δ_{max}) = (0.1, 0.4), the optimal charting parameters of the SSGR chart are (K, L) = (2.3326, 31) (see Table 1). These optimal charting parameters give $EARL_1$ = 71.87 (see Table 3). For the similar (n, δ_{min}, δ_{max}) combinations, the optimal charting parameters of the EWMA chart are (λ, J) = (0.0384, 2.3991) (see Table 2) and the associated $EARL_1$ is 29.43 (see Table 3). By setting the same $EARL_0$ value for the SSGR and EWMA control charts under comparison, a chart that yields the smallest $EARL_1$ is regarded as more efficient. For clarity, when n = 5, (δ_{min}, δ_{max}) = (0.1, 0.4), the EWMA chart has the quickest response, as it yields the lowest $EARL_1$ (= 29.43).

The comparison of the $EARL_1$ performance in Table 3 shows the superiority of the EWMA chart when (δ_{min}, δ_{max}) = (0.1, 0.4), regardless of the sample size, n. For instance, when n = 9, the EWMA chart performs best, as it has the smallest $EARL_1$ value, i.e. $EARL_1$ = 19.65, as compared to the SSGR chart, with $EARL_1$ = 41.30. Additionally, the detection ability of the EWMA chart is slightly better than the SSGR chart when n = 3 and (δ_{min}, δ_{max}) = (0.5, 0.8). For this (n, δ_{min}, δ_{max}) combination, the $EARL_1$ values for the EWMA and SSGR charts are 8.34 and 8.54, respectively. According to this comparison, the EWMA chart is more efficient for the n = 3 and (δ_{min}, δ_{max}) = (0.5, 0.8). However, the SSGR chart prevails over the EWMA chart when the sample size $n \geq 5$, (δ_{min}, δ_{max}) = (0.5, 0.8) and (δ_{min}, δ_{max}) = (0.9, 1.2), regardless of n. For example, by considering n = 7, (δ_{min}, δ_{max}) = (0.5, 0.8), the $EARL_1$ values for the SSGR chart and EWMA chart are 2.79 and 4.39, respectively. According to this comparison, the SSGR chart surpasses the EWMA chart.

Table 1 - Optimal charting parameters (K, L) of the SSGR chart with various combinations of (n, δ_{min}, δ_{max}) based on $EARL_0$ = 370.4

$(n, \delta_{min}, \delta_{max}, EARL_0)$	Side sensitive group runs chart	
	K	L
(3, 0.1, 0.4, 370.4)	2.3794	38
(3, 0.5, 0.8, 370.4)	2.0537	10
(3, 0.9, 1.2, 370.4)	1.8025	4
(5, 0.1, 0.4, 370.4)	2.3326	31
(5, 0.5, 0.8, 370.4)	1.9588	7
(5, 0.9, 1.2, 370.4)	1.7185	3
(7, 0.1, 0.4, 370.4)	2.3003	27
(7, 0.5, 0.8, 370.4)	1.8660	5
(7, 0.9, 1.2, 370.4)	1.5953	2
(9, 0.1, 0.4, 370.4)	2.2821	25
(9, 0.5, 0.8, 370.4)	1.8025	4
(9, 0.9, 1.2, 370.4)	1.5953	2

Table 2 - Optimal charting parameters (λ, J) of the EWMA chart with various combinations of (n, δ_{min}, δ_{max}) based on $EARL_0$ = 370.4

$(n, \delta_{min}, \delta_{max}, EARL_0)$	Exponentially weighted moving average chart	
	λ	J
(3, 0.1, 0.4, 370.4)	0.0270	2.2631
(3, 0.5, 0.8, 370.4)	0.1576	2.8121
(3, 0.9, 1.2, 370.4)	0.3247	2.9357
(5, 0.1, 0.4, 370.4)	0.0384	2.3991
(5, 0.5, 0.8, 370.4)	0.2275	2.8829
(5, 0.9, 1.2, 370.4)	0.4782	2.9747
(7, 0.1, 0.4, 370.4)	0.0482	2.4798
(7, 0.5, 0.8, 370.4)	0.2877	2.9195
(7, 0.9, 1.2, 370.4)	0.6140	2.9898
(9, 0.1, 0.4, 370.4)	0.05757	2.5390
(9, 0.5, 0.8, 370.4)	0.3441	2.9428
(9, 0.9, 1.2, 370.4)	0.7151	2.9954

Table 3 - EARL$_{1}$s of the SSGR chart and EWMA chart for different combinations of (n, δ_{min}, δ_{max})

	SSGR	EWMA
n	$(\delta_{min}, \delta_{max}) = (0.1, 0.4)$	
3	107.60	41.20
5	71.87	29.43
7	52.96	23.40
9	41.30	19.65
n	$(\delta_{min}, \delta_{max}) = (0.5, 0.8)$	
3	8.54	8.34
5	4.24	5.67
7	2.79	4.39
9	2.11	3.63
n	$(\delta_{min}, \delta_{max}) = (0.9, 1.2)$	
3	2.31	3.93
5	1.45	2.67
7	1.19	2.05
9	1.08	1.68

4. Conclusion

To date, control charts are widely applied in various industries. A complete understanding of a control chart's performance is important to increase the confidence of the practitioner in using it practically. The performance of the control chart is typically studied using the ARL criterion. However, the magnitude of the shift size is seldom known, and needs to be considered as a random variable. In this study, the SSGR and EWMA charts were compared using the EARL criterion. It was demonstrated that the EARL performance of the EWMA chart is better than the SSGR chart when (δ_{min}, δ_{max}) = (0.1, 0.4), regardless of the sample size. Meanwhile, the EWMA chart performance is slightly better than that of the SSGR chart when (δ_{min}, δ_{max}) = (0.5, 0.8) and small sample size, i.e. $n = 3$. When the sample size, $n \geq 5$ for (δ_{min}, δ_{max}) = (0.5, 0.8), the detection ability of the SSGR chart is better than the EWMA chart. This same phenomenon occurs when the (δ_{min}, δ_{max}) = (0.9, 1.2), regardless of the sample size. This study contributes to practitioners in the selection of the control charts, when the next shift size cannot be determined in advance. Future research work can be considered to compare the performance of the control charts using more than one performance criterion.

Acknowledgement

This research was supported by the Universiti Kebangsaan Malaysia, Geran Galakan Penyelidikan, GGP-2020-040.

References

[1] Kasmin, A., Masood, I., Abdul Rahman, N., Abdul Kadir, A., & Abdol Rahman, M. (2020). Control Chart Pattern Recognition Using Small Window Size for Identifying Bivariate Process Mean Shifts. International Journal of Integrated Engineering, 13(2), 208-213.

[2] Kasmin, A., Masood, I., Abdul Rahman, N., Fadzillah, A., & Abdol Rahman, M. (2020). Identifying Unnatural Variation in Precision Rotational Part Manufacturing. International Journal of Integrated Engineering, 13(2), 189-195.

[3] Montgomery, D. C. (2019). Introduction to Statistical Quality Control (New York: John Wiley & Sons, Inc) 8th ed.

[4] Kamrisham, M. F., Mustafa, Z., & You, H. W. (2019). Satu kajian terhadap carta kawalan pensampelan berganda sintetik dan larian kumpulan sensitif sampingan. Undergraduate Research Journal for Mathematical Sciences and Data Science 3, 250-256.

[5] Abbasi, S. A. (2020). Efficient control charts for monitoring process CV using auxiliary information. IEEE Access, 46176-46192.

[6] You, H. W., Khoo, M. B. C., Lee, M. H., Castagliola, P., & Saha, S. (2017). Optimal design of exponentially weighted moving average-chart for the mean with estimated process parameters. Communications in Statistics - Theory and Methods, 46(22), 11077-11090. DOI: 10.1080/03610926.2016.1257716.

[7] Chong, Z. L., Khoo, M. B. C., Teoh, W. L., You, H. W., & Castagliola, P. (2018). Optimal design of the side-sensitive modified group runs (SSMGR) $\bar{X}$ chart when process parameters are estimated. Quality and Reliability Engineering International, 35, 246-262. DOI: 10.1002/qre.2395.

[8] You, H. W., Khoo, M. B. C., Chong, Z. L., & Teoh, W. L. (2020). The expected average run length of the EWMA median chart with estimated process parameters. Austrian Journal of Statistics, 49, 19-24. doi:10.17713/ajs. v49i3.1020.

[9] Gadre, M. P., & Rattihalli, R. N. (2007). A side sensitive group runs control chart for detecting shifts in the process mean. Statistical Methods and Applications, 16, 27-37.

[10] You, H. W., Khoo, M. B. C., Castagliola, P., & Ou, Y. (2015). Side sensitive group runs $\bar{X}$ chart with estimated process parameters. , Computational Statistics, 30, 1245-1278.

[11] Roberts, S. W. (1959). Control chart test based on geometric moving averages. Technometrics, 1(3), 239-250.

[12] Shahrin, M. A., Mustafa, Z., & You, H. W. (2019). Penilaian terhadap carta kawalan median purata bergerak berpemberat eksponen. Undergraduate Research Journal for Mathematical Sciences and Data Science 3, 257-263.