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Monte Carlo Simulation of a Beam Resting on an Elastic Foundation Considering the Two-Dimensional Stochastic Properties of the Elastic Modulus

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Abstract

The analysis of the random behavior of beams on an elastic foundation, considering a two-dimensional random elastic modulus, contributes to bringing the analytical model closer to the physical model of the problem and enhancing the reliability of structural calculations. This paper aims to develop a Monte Carlo simulation (MCs) to represent the two-dimensional random field of elastic modulus combined with the finite element method to analyze the random response of beams resting on an elastic foundation according to the Winkler model. The spectral representation method generates the two-dimensional elastic modulus's Gaussian. This sample function is used to construct the formulation of finite elements. The influence of the random field's standard deviation, the correlation distance along the in-plane axes, and the stiffness of the elastic foundation on the coefficient of variation of displacement are also investigated and analyzed in detail in this article. The two-dimensional randomness of the elastic modulus and the stiffness coefficient of the foundation significantly affect the random response of the beam. The coefficient of variation *(COV)* of displacement tends to increase when the standard deviation of the stochastic field or the correlation distance along the axes increases. Still, conversely, when the stiffness of the elastic foundation rises, the coefficient of variation decreases. The *COV* of displacement approaches the standard deviation value of the stochastic field of material properties when the correlation distance along the axes approaches infinity.

1. Introduction

The finite element method (FEM) is commonly widely used in structural analysis due to its ability to handle complex geometries, various boundary conditions, and structures, including multiple materials [1-3]. In structural analysis, factors such as loads, effects, or material properties often exhibit randomness or uncertainty [4-7]. However, the deterministic nature of FEM limits its capacity to account for structural randomness resulting from stochastic input factors or uncertain parameters. To overcome this limitation, the stochastic finite element method (SFEM) combines standard FEM with stochastic computation theories typically used in stochastic structural analysis [8-11]. SFEM involves two main stages for analysing structures with stochastic input factors or uncertain parameters. Initially, these factors are represented as fields or variables within the computational framework.

Subsequently, random features or unknown parameters are introduced into the system using SFEM techniques [12]. Several SFEM techniques have been developed to facilitate stochastic structural analysis, including Monte Carlo simulation [13], spectral approaches [14], and perturbation methods [15]. These methods enhance the predictive accuracy and reliability of FEM by accommodating the inherent stochasticity of real-world structural scenarios.

The MCs involve generating sample functions from an input random field and assessing structural behavior based on these samples to derive a set of responses. A statistical analysis of these responses yields the statistical characteristics of the structural response. Shinozuka and Jan [16] pioneered the application of MCs in stochastic mechanics for the numerical simulation of stationary random fields. Despite its reliability, the Monte Carlo simulation method demands substantial computational resources, particularly when tackling complex problems with higher dimensions, leading to increased computational time and cost. Numerous strategies have been developed to reduce the number of simulation samples to address this, thereby mitigating computational expenses and time consumption [17]. Effective implementation of MCs necessitates computations using sufficient samples to ascertain the statistical properties or failure probabilities of structures. Despite the computational demands, this approach remains widely employed for stochastic analysis in various systems.

The scientific community has paid considerable attention to the stochastic analysis of beam structures with random factors such as loads and impacts or uncertain parameters such as material properties or geometric dimensions. Vanmarcke and Grigoriu [18] used random finite elements to analyze beam structures with randomly varying stiffness. The stiffness of the beam is assumed to be a random field that varies unidirectionally along the length of the beam. The statistical characteristics of beam displacements, such as expectation and variance, are established. Deodatis [19] introduced a weighted integration method for discretizing a random field by calculating integrals of monomials with the stochastic field over the element domain. This method discretizes the random field into random variables on the element domain and can be applied to the same element mesh for any correlation distance. It simultaneously represents the displacement and random fields using the element shape function. The random stiffness matrix of the frame element with an elastic modulus randomly varying in one direction along the element length is also established by applying the concepts of virtual work and potential energy [20]. Sharing the same idea of the weighted integration method, Takada [21] also used the weighted integration method to analyze the random response of bar structures. Hien [22] developed a weighted integration technique to discretize random fields combined with SFEM to analyze the stability of columns with cross-sections varying along their length. Deodatis et al. [23] proposed a weighted integration method for two-dimensional stochastic fields combined with the SFEM developed from the weighted integration method for one-dimensional random fields.. Ren et al. [24] proposed a variational method to directly determine the mean function and displacement covariance of beams with one-dimensional randomly varying stiffness without depending on the first derivatives of displacements or the widely used perturbation method. This proposed method can be applied in the case of large random stiffness deviations. Adhikaria and Mukherjee [25] used the Castigliano method [26] to construct the element stiffness matrix in two cases: beams with varying elastic modulus or beams with randomly varying stiffness along the length of the beam. Unlike the conventional finite element method, the displacement field is assumed through the nodal shape and displacement function. The Castigliano method builds the element's stiffness matrix from the relationship between force and displacement. The random stiffness matrix is divided into two parts, including the predetermined part and the random part. A beam element contains three random variables discretized from the random field. Material properties do not only change randomly along the beam length but can also change randomly in the remaining directions [27]. The averaging method over the element domain represents the elastic modulus random field. Then, the random field on each element is approximated as a random variable. At the same time, the perturbation method and the Newman technique of series expansion combined with MCs will be used for stochastic analysis.

The Winkler and Pasternak foundation models are widely used for simulating the interaction between structural elements and elastic foundations. The Winkler model, characterized by independent springs, offers simplicity and computational efficiency, making it suitable for problems with straightforward boundary conditions and localized load applications [28, 29]. However, its primary limitation lies in neglecting the interaction between adjacent points on the foundation, which can lead to inaccuracies in predicting the behavior of continuous or distributed systems. In contrast, the Pasternak model incorporates a shear layer to account for interactions between neighboring points, enhancing the accuracy for problems involving continuous foundations or complex loading scenarios [30, 31]. Despite its advantages, the Pasternak model introduces additional complexity in parameter determination and computational implementation due to the inclusion of the shear modulus. For the present study on beam analysis, the Winkler foundation model is selected due to its simplicity and adequacy in capturing the essential behavior of the system under consideration. The assumptions of localized load distribution and negligible interaction effects between foundation points are reasonable for this problem, ensuring that the Winkler model provides a balance between accuracy and computational efficiency [32].

Previous research frequently assumes that the elastic modulus of the beam varies randomly across its length.Very few studies analyse beams whose elastic modulus varies randomly along beam length and height directions. Therefore, in this paper, stochastic analysis of continuous beams on elastic foundations with two-dimensional random field in elastic modulus is employed by combing the MCs and the finite element method. The effect of a stochastic field, such as the correlation distance (*CD*) and the standard deviation (*SD*) of the elastic modulus stochastic field, and the stiffness of the elastic foundation on the COV of displacement is investigated in detail.

2. Stochastic Static Response Analysis of Beams Using MCs

Consider the beam construction depicted in Fig. 1, which has a rectangular cross-section. The geometric dimensions of the beam include the following span length: *L*, cross-sectional height: *h*, and cross-sectional width: *b*. The global coordinate system of the entire structure is selected, including the *x*-axis along the length of the span and the *y*-axis along the height of the cross-section. The beam is resting on an elastic foundation. The Winkler elastic foundation model, which has a stiffness coefficient of K_F, was the elastic foundation model employed in this investigation. The elastic modulus of the beam *E(x, y)* is assumed to vary randomly along the length and height of the beam as Eq. (1).

$$E(x,y) = E_0\left[1 + f(x,y)\right] \tag{1}$$

here, E_0 reflects the elastic modulus's mean values. *f(x, y)* is a stationary, Gaussian random field. This random field changes bidirectionally along the *x* and *y* axes. The expected value of random field is zero. The stochastic field of elastic modulus is described as the sum of the cosine series of the uniform random variable [33].

$$f(x,y) = \sqrt{2}\sum_{p_1=0}^{P_1-1}\sum_{p_2=0}^{P_2-1}\left[A_{p_1p_2}\cos\left(\Upsilon_{1,p_1}x + \Upsilon_{2,p_2}y + \Phi^{(1)}_{p_1p_2}\right) + B_{p_1p_2}\cos\left(\Upsilon_{1,p_1}x - \Upsilon_{2,p_2}y + \Phi^{(2)}_{p_1p_2}\right)\right] \tag{2}$$

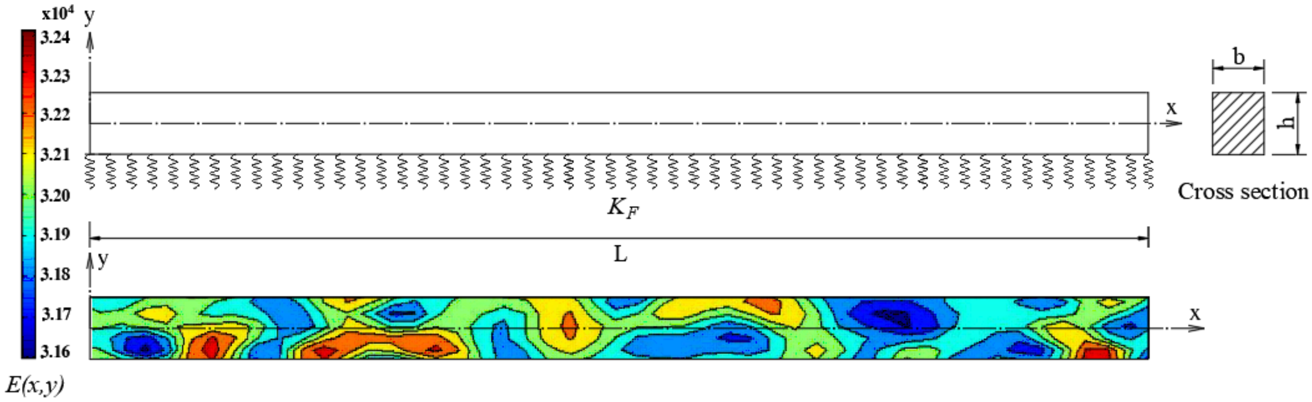


Fig. 1 *Randomly varying two-dimensional elastic modulus beam*

The parameters in Eq. (2) are determined according to the following formulas:

$$A_{p_1p_2} = \sqrt{2S_{ff}\left(\Upsilon_{1,p_1},\Upsilon_{2,p_2}\right)\Delta\Upsilon_1\Delta\Upsilon_2}; \qquad B_{p_1p_2} = \sqrt{2S_{ff}\left(\Upsilon_{1,p_1},-\Upsilon_{2,p_2}\right)\Delta\Upsilon_1\Delta\Upsilon_2} \tag{3}$$

$$\Upsilon_{1,p_1} = p_1\Delta\Upsilon_1; \quad \Upsilon_{2,p_2} = p_2\Delta\Upsilon_2 \tag{4}$$

$$\Delta\Upsilon_1 = \frac{\Upsilon_{1u}}{P_1} \quad ; \quad \Delta\Upsilon_2 = \frac{\Upsilon_{2u}}{P_2} \tag{5}$$

$$p_1 = \{0,1,...,P_1-1\} \qquad p_2 = \{0,1,,...,P_2-1\} \tag{6}$$

$\Phi^{(1)}_{p_1p_2}$ and $\Phi^{(2)}_{p_1p_2}$ are the uniform random phase angle within the scope of $[0,2\pi]$. The power spectral density function (PSDF) used in Eq. (7) is given as:

$$S_{ff}\left(\Upsilon_1,\Upsilon_2\right)=\sigma_f^2\frac{d_x d_y}{4\pi}\exp\left[-\left(\frac{d_x\Upsilon_1}{2}\right)^2-\left(\frac{d_y\Upsilon_2}{2}\right)^2\right] \tag{7}$$

In this Eq. (5), $\Upsilon_{1u},\Upsilon_{2u}$ are the upper cut-off limit of wave numbers in the PSDF can be determined by Eq. (8). In cases where determining $\Upsilon_{1u},\Upsilon_{2u}$ is mathematically challenging, the value of η can be assumed to be so small that η = 0.08% can be chosen.

$$\int_0^{\Upsilon_{1u}}\int_{-\Upsilon_{2u}}^{\Upsilon_{2u}} S_{ff}\left(\Upsilon_1,\Upsilon_2\right)d\Upsilon_1 d\Upsilon_2=(1-\eta)\int_0^{\infty}\int_{-\infty}^{\infty} S_{ff}\left(\Upsilon_1,\Upsilon_2\right)d\Upsilon_1 d\Upsilon_2 \tag{8}$$

In the finite element method context, the beam is divided into N_e elements. The elastic modulus of the element varies randomly in two dimensions according to the length and height of the element, as shown in Eq. (9)

$$E_e(x,y)=E_0\left[1+f_e(x,y)\right] \tag{9}$$

Substituting Eq. (2) into Eq. (9), the elastic modulus of the element is obtained as Eq. (10), where Δ_e is the distance from the end of the beam to the *e-th* element.

$$E_e=E_0\left[1+\sqrt{2}\sum_{p_1=0}^{P_1-1}\sum_{p_2=0}^{P_2-1}\left\{\begin{array}{l}A_{p_1p_2}\cos\left[\Upsilon_{1,p_1}\left(x+\Delta_e\right)+\Upsilon_{2,p_2}y+\Phi_{p_1p_2}^{(1)}\right]+\\ B_{p_1p_2}\cos\left[\Upsilon_{1,p_1}\left(x+\Delta_e\right)-\Upsilon_{2,p_2}y+\Phi_{p_1p_2}^{(2)}\right]\end{array}\right\}\right] \tag{10}$$

Consider a beam element with two nodes, each having two degrees of freedom. The length of the element is *a*. The displacement of the beam element is determined based on the element's shape function and the element's nodal displacement as Eq. (11). In equation Eq. (11), $w_e(x)$ is the displacement of the beam element, and $\{q\}_e$ is the nodal displacement vector of the beam element.

$$w_e(x)=\left[H_1(x)\quad H_2(x)\quad H_3(x)\quad H_4(x)\right]\begin{Bmatrix}q_1\\q_2\\q_3\\q_4\end{Bmatrix}=\left[H_i(x)\right]\{q\}_e \tag{11}$$

$\left[H_i(x)\right]$ is the shape function matrix of the bending beam element determined by the following formulas:

$$\begin{aligned}H_1(x)&=1-3\frac{x^2}{a^2}+2\frac{x^3}{a^3}\\ H_2(x)&=x\left(1-2\frac{x}{a}+\frac{x^2}{a^2}\right)\\ H_3(x)&=3\frac{x^2}{a^2}-2\frac{x^3}{a^3}\\ H_4(x)&=x\left(-\frac{x}{a}+\frac{x^2}{a^2}\right)\end{aligned} \tag{12}$$

Expression of strain energy U_e for bending is determined as Eq. (13):

$$U_e=\frac{1}{2}\int_{V_e}E_e\left[y\frac{\partial^2 w_e(x)}{\partial x^2}\right]^2 dV_e \tag{13}$$

The elastic foundation's potential energy T_F^e is written as the following equation:

$$T_F^e = \int_{V_e} K_F \left(\frac{w_e}{2}\right)^2 dV_e \tag{14}$$

The stiffness matrix of the beam element is found using the virtual work principle, where $[B]$ is the matrix that defines the relationship between deformation and displacement.

$$[K]_e = \int_{V_e} [B]^T E_e [B] dV_e + \int_{V_e} \left[H_i(x)\right]^T K_F \left[H_i(x)\right] dV_e \tag{15}$$

with

$$[B] = -y\left[\left(-\frac{6}{a^2}+\frac{12x}{a^3}\right) \; \left(-\frac{4}{a}+\frac{6x}{a^2}\right) \; \left(\frac{6}{a^2}-\frac{12x}{a^3}\right) \; \left(-\frac{2}{a}+\frac{6x}{a^2}\right)\right] \tag{16}$$

Substituting Eq. (10) into Eq. (15), the following result is obtained:

$$\begin{aligned}[K]^e &= \underbrace{\int_{V_e} [B]^T E_0 [B] dV_e}_{[K_0]^e} + \underbrace{\int_{V_e} \left[H_i(x)\right]^T K_F \left[H_i(x)\right] dV_e}_{[K_{0F}]^e} \\ &+ \underbrace{\int_{V_e} [B]^T E_0 \sqrt{2} \sum_{p_1=0}^{P_1-1} \sum_{p_2=0}^{P_2-1} \begin{Bmatrix} A_{p_1p_2} \cos\left[\Upsilon_{1,p_1}(x+\Delta_e) + \Upsilon_{2,p_2} y + \Phi_{p_1p_2}^{(1)}\right] + \\ B_{p_1p_2} \cos\left[\Upsilon_{1,p_1}(x+\Delta_e) - \Upsilon_{2,p_2} y + \Phi_{p_1p_2}^{(2)}\right] \end{Bmatrix} [B] dV_e}_{[\Delta K]^e}\end{aligned} \tag{17}$$

with

$$[K_0]^e = \int_{V_e} [B]^T E_0 [B] dV_e = \frac{E_0 I}{a^3} \begin{bmatrix} 12 & 6a & -12 & 6a \\ & 4a^2 & -6a & 2a^2 \\ & & 12 & -6a \\ sym & & & 4a^2 \end{bmatrix} \tag{18}$$

$$[K_{0F}]^e = \int_{V_e} \left[H_i(x)\right]^T K_F \left[H_i(x)\right] dV_e = K_F b \begin{bmatrix} \frac{13}{35}a & \frac{11}{210}a^2 & \frac{9}{70}a & -\frac{13}{420}a^2 \\ & \frac{1}{105}a^3 & \frac{13}{420}a^2 & -\frac{1}{140}a^3 \\ & & \frac{13}{35}a & -\frac{11}{210}a^2 \\ sym & & & \frac{1}{105}a^3 \end{bmatrix} \tag{19}$$

$$[\Delta K]^e = \int_{V_e} [B]^T E_0 \sqrt{2} \sum_{p_1=0}^{P_1-1} \sum_{p_2=0}^{P_2-1} \begin{Bmatrix} A_{p_1p_2} \cos\left[\Upsilon_{1,p_1}(x+\Delta_e) + \Upsilon_{2,p_2} y + \Phi_{p_1p_2}^{(1)}\right] + \\ B_{p_1p_2} \cos\left[\Upsilon_{1,p_1}(x+\Delta_e) - \Upsilon_{2,p_2} y + \Phi_{p_1p_2}^{(2)}\right] \end{Bmatrix} [B] dV_e \tag{20}$$

The global stiffness matrix is determined from the stiffness matrix of the element as follows:

$$[K] = \bigcup_{e=1}^{N_e} [K_0]^e + \bigcup_{e=1}^{N_e} [K_{0F}]^e + \bigcup_{e=1}^{N_e} [\Delta K]^e \tag{21}$$

The nodal displacement vector $\{U\}$ is defined as Eq. (22) , where $[K]$ is the global stiffness matrix and $\{P\}$ is the load vector.

$$[K]\{U\} = \{P\} \tag{22}$$

Using Eq. (10) produces *N* samples of the two-dimensional random field of the elastic modulus. The global stiffness matrix of the entire structure at trial *i-th* is determined as Eq. (21). Analysis of the static response of the structure on the elastic modulus sample set is created using Eq. (22). From then, *N* nodal displacement vectors of the structure are obtained. Statistical processing on the obtained data set can determine the statistical characteristics of the sample, such as the expectation of the displacement as Eq. (23) and the variance of the displacement as Eq . (24).

$$\mu[U] = \frac{1}{N}\sum_{i=1}^{N} U^i \tag{23}$$

$$Var[U] = \frac{1}{N-1}\left(\sum_{i=1}^{N}\left(U^i\right)^2 - N\left(\mu[U]\right)^2\right) \tag{24}$$

The coefficient of variation is a statistical characteristic used to compare the relative variability of different data sets. It is defined in Eq. (25) as the ratio between the sample's standard deviation and sample expectations.

$$COV = \frac{\sqrt{Var[U]}}{\left|\mu[U]\right|} \tag{25}$$

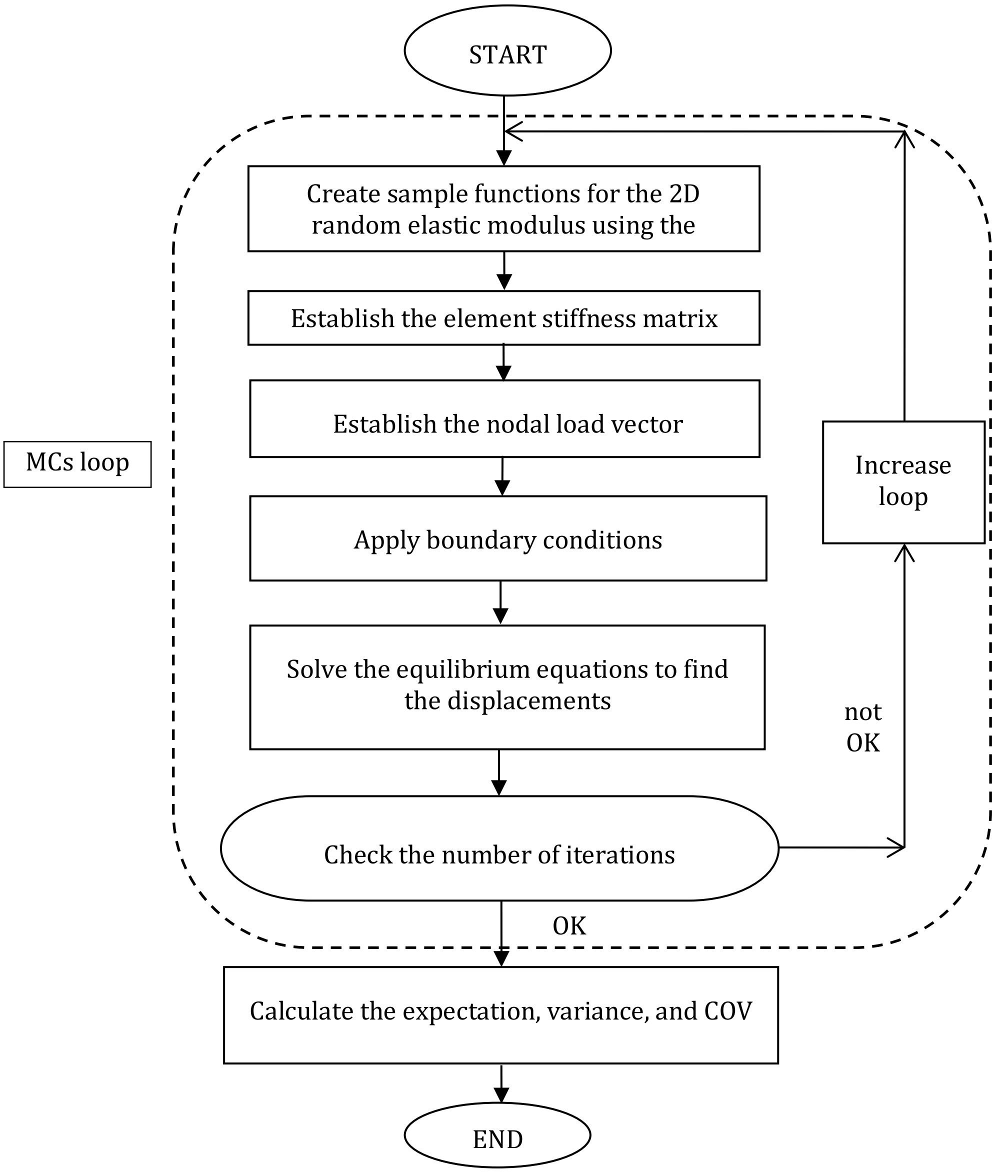


Fig. 2 *Block diagram of random behavior analysis of a beam using MCs*

Figure 2 illustrates the block diagram of the steps required to analyze the random behavior of a beam with a two-dimensional random elastic modulus. The beam is placed on an elastic foundation modeled as a Winkler foundation. In this process, sample functions of the two-dimensional random field are generated using the spectral representation method, combined with the standard FEM to construct characteristic matrices of the system, such as the global stiffness matrix and load vector. The statistical characteristics of the beam's displacements are determined through conventional statistical analysis tools.

3. Numerical Examples and Discussion

3.1 Verification of Computational Results

To verify the displacement analysis results of the beam obtained through the proposed finite element method (FEM) in this study, an analytical approach [34] was employed to analyze the static response of the beam. The differential equation describing the displacement profile of a beam under a distributed load *q(x)* is given by Eq. (26).

$$EI\frac{d^4w}{dx^4}+K_Fbw=bq(x) \quad (26)$$

To solve Eq. (26) for the case of a beam subjected to a uniformly distributed load q_0, the displacement of the beam is obtained as shown in Eq. (27), where C_1, C_2, C_3, and C_4 are the constants of integration determined based on the boundary conditions.

$$w = e^{\lambda x}\left(C_1 \cos \lambda x + C_2 \sin \lambda x\right) + e^{-\lambda x}\left(C_3 \cos \lambda x + C_4 \sin \lambda x\right) + \frac{q_0}{K_F} \tag{27}$$

with

$$\lambda = \sqrt[4]{\frac{K_F}{EI}} \tag{28}$$

Figure 3 illustrates the error in mid-span displacement of the beam determined using the proposed FEM compared to the analytical solution, corresponding to an increasing number of elements ranging from *2* to *16*. The beam, with a rectangular cross-section and simply supported at both ends, has the following geometric dimensions: *L = 10 m,b = 0.4 m,h = 0.6 m*. The material properties include a modulus of elasticity $E=30\times10^3$ Mpa and a foundation stiffness coefficient $K_F=10^5$. The beam is subjected to a uniformly distributed load $q_0=10$ *kN/m*. It is evident that as the number of elements increases, the beam's displacement converges towards the displacement value obtained through the analytical method. Therefore, the proposed FEM solution in this study is both suitable and reliable for analyzing the beam's behavior.

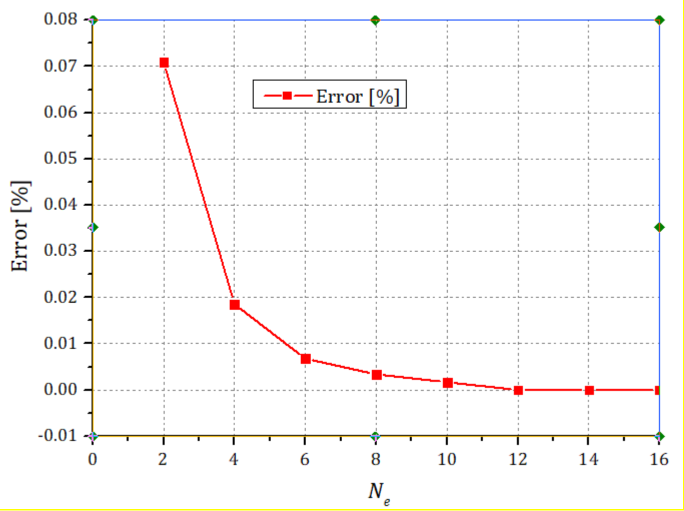


Fig. 3 *Error in displacement determination using FEM compared to the analytical method*

3.2 Analysis of the Influence of Factors in the 2D Random Field on the COV

Consider a beam with a rectangular cross-section on an elastic foundation with a stiffness coefficient $K_F = 10^5$ *Mpa*. Elastic modulus $E_0 = 32\times10^3$ *MPa*, width *b = 0.3 m*, height *h = 0.6 m*, and length *L = 8 m* are the geometrical and material properties of the beam. A load *Q = 50 kN* is applied to the beam at the middle span. With the following equations for the power spectral density function and autocorrelation function, the elastic modulus is considered a stationary, univariate, two-dimensional random field.

$$R_{ff}\left(x_2 - x_1, y_2 - y_1\right) = \sigma_f^2 \exp\left[-\left(\frac{x_2 - x_1}{d_x}\right)^2 - \left(\frac{y_2 - y_1}{d_y}\right)^2\right] \tag{26}$$

$$S_{ff}\left(\Upsilon_1, \Upsilon_2\right) = \sigma_f^2 \frac{d_x d_y}{4\pi} \exp\left[-\left(\frac{d_x \Upsilon_1}{2}\right)^2 - \left(\frac{d_y \Upsilon_2}{2}\right)^2\right] \tag{27}$$

Where, σ_f is the standard deviation of the elastic modulus random field, d_x and d_y are the correlation distances along the *x*-axis and *y*-axis. Ten thousand samples of the elastic modulus of the beam were generated using Eq. (10). Fig. 4 is a function sample of the elastic modulus of the beam in one trial. Analyzing the structure's response based on the sample set of elastic modulus is created to obtain the response set of the structure. Fig. 5 and Fig. 6 are frequency histograms of nodal displacements at the mid-span of the beam corresponding to different correlation distances. The standard deviation of the random field is assumed to be *0.05*. As the correlation distance along the axes increases, the displacement of the beam obtained in the *i-th* trial becomes more and more dispersed from the average displacement value. In other words, as the correlation distance increases, the standard deviation of displacement also increases. The dependence of the displacement COV on the correlation length in the two-dimensional random field presented in this study is consistent with the analyses reported in [20], where the authors developed the weighted integral method combined with perturbation techniques to perform random analysis of bar structures with one-dimensional random elastic modulus.

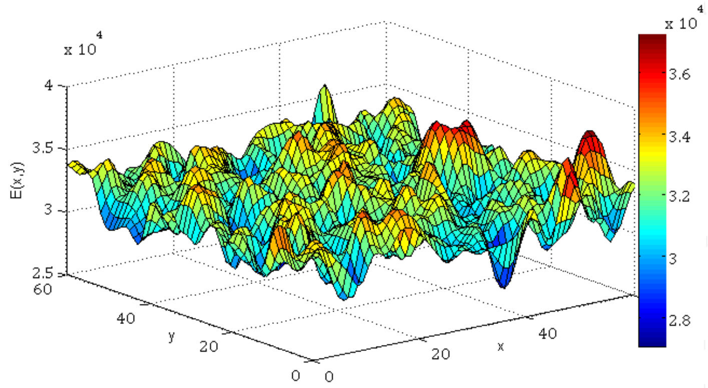
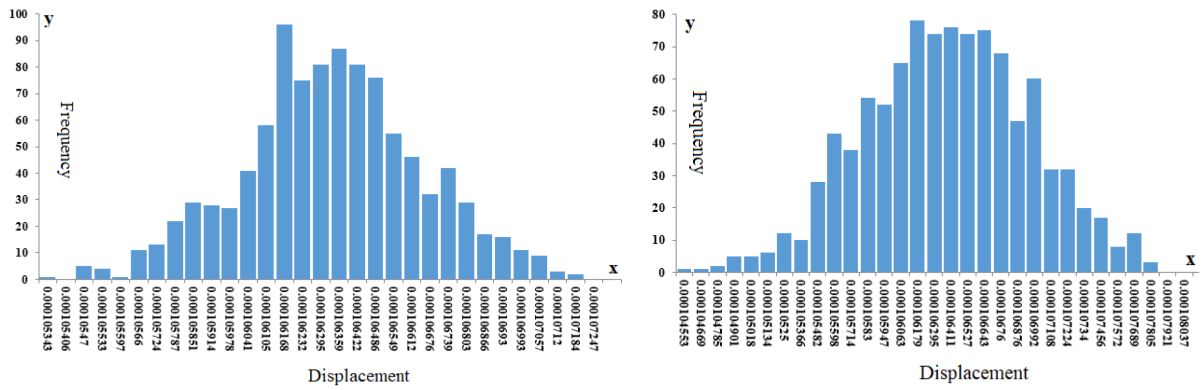
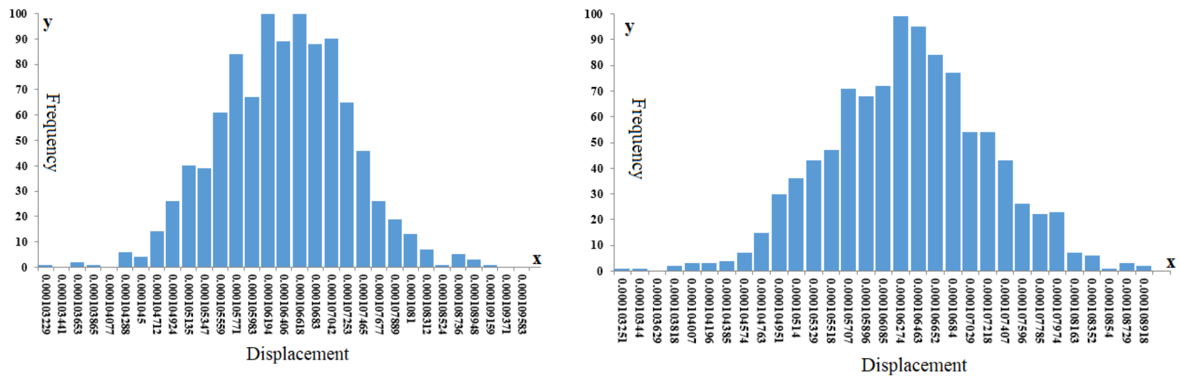


Fig. 4 *A sample function of the stochastic field*



(a) **(b)**

Fig. 5 *Frequency histogram of displacement (a) $d_x = d_y = 0.005L$; (b) $d_x = d_y = 0.1L$*



(a) **(b)**

Fig. 6 *Frequency histogram of displacement (a) $d_x = d_y = L$; (b) $d_x = d_y = 50L$*

The influence of the correlation distance along the coordinate axes on the coefficient of variation of displacement is shown in Fig. 7. We can observe that the coefficient of variation increases as the correlation distance increases. The increase in the coefficient of variation corresponds to different values of the correlation distance. The *COV* of displacement rises sharply when the correlation distance is small, and the *COV* continues to increase when the correlation distance is considerable, but this increase is not significant. In addition, it can also be observed that the influence of the axial correlation distance along the span length is more significant than the influence of the axial correlation distance along the section's height.

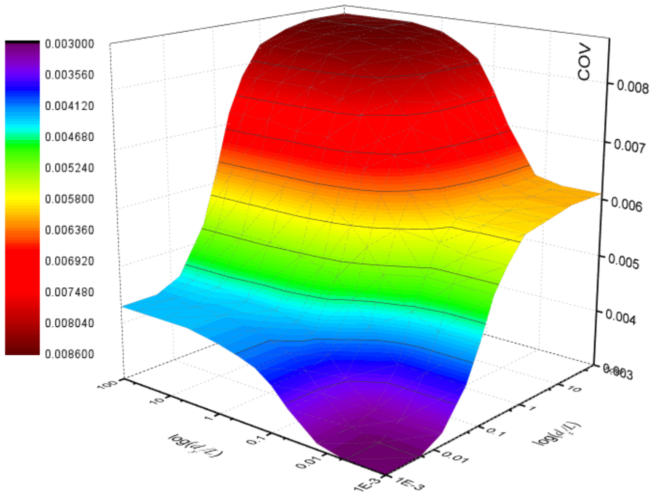


Fig. 7 *The influence of CD on COV of displacement*

Fig. 8 shows the *COV* of response versus the *CD* at the different *SD* of a stochastic field. The correlation distance $d_x = d_y = d$ varies from *0.001L* to *100L*. The *SD* of the elastic modulus random field ranges from *0.05* to *0.20*. In all cases, the *COV* of displacements tends to be similar, starting from small values for the slight standard deviation of the elastic modulus random field up to large values. The *COV* of displacement increases as the correlation distance increases or the standard deviation of the random field increases. At large values of correlation distance, as the standard deviation of the random field increases, the *COV* increases almost equally for different correlation distances. In the analysis of the free vibration problem of a beam with a one-dimensional random elastic modulus [11] using SFEM, the authors demonstrated a positive correlation between the COV of natural frequencies and the *SD* of the one-dimensional random elastic modulus. This is analogous to the positive correlation between the COV of displacements and the SD of the two-dimensional random elastic modulus observed in this study.

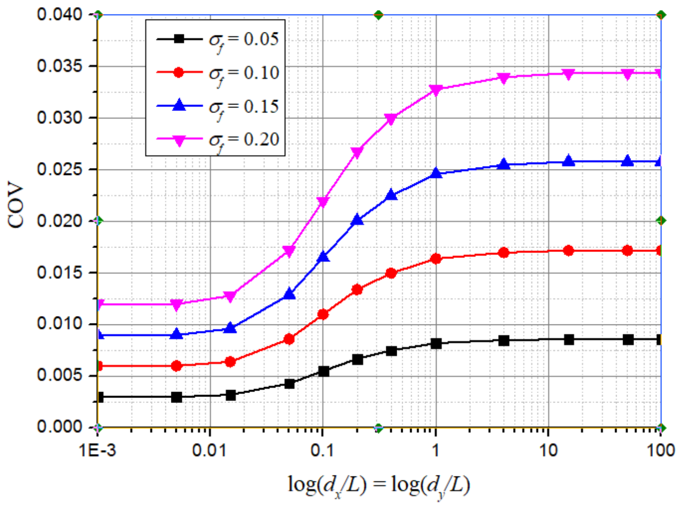


Fig. 8 *COV of displacement as a function of the CD for the different SD of a random field*

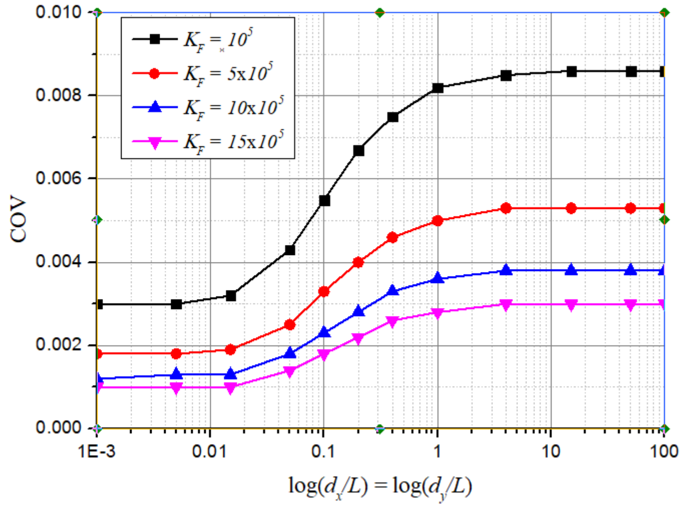


Fig. 9 *The influence of the foundation stiffness coefficient on COV*

The foundation stiffness coefficient, corresponding to the values $K_F = 10^5$ *Mpa*, $K_F = 5\times10^5$ *Mpa*, $K_F = 10\times10^5$ *Mpa*, and $K_F = 15\times10^5$ *Mpa*, is evaluated to analyze its influence on the *COV* of beam displacement. In Fig. 9, the impact of the foundation stiffness coefficient on the coefficient of variation of beam displacement is illustrated. There is a clear influence of the foundation stiffness coefficient on the *COV* of beam displacement. Across different values of the foundation stiffness coefficient, the *COV* of beam displacement appears to be consistently similar. The *COV* decreases as the foundation stiffness coefficient increases, particularly evident at larger correlation distance values.

4. Conclusions

The paper successfully developed the random finite element method combined with Monte Carlo simulation to analyze the response variability of a beam resting on a Winkler foundation. The elastic modulus of the beam is modeled as a two-dimensional, stationary, univariate Gaussian random field. The random fields of the elastic modulus are simulated as sample functions using the spectral representation method, expressed as a sum of a series of cosine functions. The analysis results indicate that the two-dimensional randomness of the elastic modulus and the stiffness coefficient of the foundation significantly affect the static random response of the beam. The *COV* of the displacement shows a positive correlation with the correlation length or the standard deviation of the two-dimensional random field. The influence of the correlation length on the *COV* varies across different magnitudes of the correlation length. Additionally, the effect of the foundation stiffness coefficient on the COV is evident. As the foundation stiffness coefficient increases, the *COV* of the beam displacement decreases, and this effect becomes more pronounced at larger correlation lengths.

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Conflict of Interest

Authors declare that there is no conflict of interests regarding the publication of the paper.

Author Contribution

The authors confirm contribution to the paper as follows: ***study conception and design:*** *Nguyen Dang Diem, Ta Duy Hien;* ***data collection:*** *Nguyen Dang Diem, Ta Duy Hien;* ***analysis and interpretation of results:*** *Nguyen Dang Diem, Ta Duy Hien;* ***draft manuscript preparation:*** *Nguyen Dang Diem. All authors reviewed the results and approved the final version of the manuscript.*

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