

Comparison Between Runge-Kutta Method, Euler Method and Modified Euler Method for Difference Order of Ordinary Differential Equation

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Abstract

Ordinary Differential Equations (ODEs), along with Initial Value Problems (IVPs), are fundamental tools in various fields such as engineering, physics, and mathematics, where they are used to model dynamic systems and processes. Issues related to differential equations frequently arise in various scientific and engineering fields. In this research, the primary focus is on comparing the performance of three numerical methods, which are Euler's Method, the Modified Euler Method, and the Runge-Kutta Method, in solving ODEs with IVPs. These methods are assessed by solving ODEs of different orders (first, second, and third order), with the equations first being transformed into systems of first-order differential equations. The goal is to identify the most accurate method for solving such problems. To achieve this, the study calculates the errors associated with each method and compares them to determine which one produces the least error. The results from this analysis demonstrate that the Modified Euler Method consistently yields better accuracy than both the standard Euler Method and the Runge-Kutta Method when applied to ODEs of varying orders.

1. Introduction

Numerical methods for solving initial value problems include the Euler method, the Taylor series method, and the Runge-Kutta method, which are employed to approximate mathematical procedures. The approximation is necessary due to the inability to solve the procedure analytically, as the analytical method is intractable. Analytical methods are applicable to a limited set of differential equations and are not feasible for a broad range of such problems [1]. This paper examines the Euler and Runge-Kutta methods, incorporates the Modified Euler Method, and subsequently compares them to determine the most effective approach for solving the initial value problem by numerical computation [2].

Initial Value Problem (IVP) is defined as a problem where to find a solution to a differential equation that will satisfy a given Initial Value, $y(x_0) = y_0$. In multivariable calculus [3], an initial value problem consists of an ordinary differential equation (ODE) and an initial condition that specify the value of the unknown function at a specific location in the domain. Modeling a system in physics or other sciences generally requires addressing an initial value issue. In that context, the differential initial value refers to an equation that defines how the system changes over time given the initial circumstances of the problem.

Ordinary Differential Equation (ODE) is an equation that contains only one independent variable and one or more of its derivatives with respect to the variable, or, in other words, it has one independent variable, x , and the real dependent variable, y , with some of its derivatives to represent the relation [4]. Order differential equations are categorized by their order, which is determined by the highest derivative that appears in the equation.

The term 'numerical computations' refers to the use of computers to solve real-world problems. This problem-solving process involves several distinct phases [5]. The first stage is formulation, where scientists plan to solve an issue on a computer by developing a mathematical model of a physical event. During this phase, they define objectives, input data, checks, and the type and amount of output needed. Once an issue is formulated, numerical methods and preliminary error analysis are developed to solve it. An algorithm, a full and unambiguous set of processes used to solve a mathematical problem, is then chosen. Numerical analysis assists in selecting or creating appropriate algorithms. The algorithm or group of algorithms chosen must be suitable for solving the problem. Additionally, numerical analysis must evaluate all potential sources of error that could affect the results, considering the required level of accuracy. This involves estimating the magnitude of round-off and discretization errors and then choosing an appropriate step size or number of iterations. Therefore, the goal of this article is to determine which numerical method, either the Euler Method or the Runge-Kutta Method, is more accurate for solving initial value problems. The study also aims to investigate the effect of the order of the ordinary differential equation on the accuracy of both methods.

2. Literature Review

Lanlege et al. [6] performed a comparative analysis of the Euler and Runge-Kutta methods for ordinary differential equations of second and fourth order. The research involved solving two differential equations utilizing the Euler method of second and fourth orders, alongside the Runge-Kutta method of second and fourth orders. The solutions will be compared with the actual solutions to assess the errors incurred and determine which strategy produced superior accuracy. The objective was to create efficient numerical methods that maintain the same order of accuracy as the original initial values for exact solutions to fourth-order differential equations, without transforming them into a system of first-order differential equations [6]. The Runge-Kutta method and its stages were utilized to solve the differential systems derived from the approximate solutions to the problem. The comparison of the methodologies demonstrated that the Euler approach produced more accurate approximate findings than the Runge-Kutta method. Additional study was performed to determine the formula with more precision.

Zhang & Guan [7] conducted a study comparing various numerical algorithms for solving initial value problems in ordinary differential equations. The authors employed MATLAB software to execute the Euler technique, the modified Euler method, and the classical Runge-Kutta method for addressing initial value problems of ordinary differential equations. The numerical study revealed that the Euler technique demonstrated greater errors due to its simplicity, while the modified Euler approach presented a heightened computing burden. Conversely, the classical Runge-Kutta approach exhibited exceptional accuracy and stability [7]. The study concentrated on first-order differential equations executed in MATLAB. The research provided comprehensive code for executing the first-order differential equations in MATLAB. The study determined that the modified Euler technique provided superior numerical accuracy relative to the Euler method, whereas the fourth-order Runge-Kutta method surpassed both the Euler and modified Euler methods. Furthermore, regarding computational effort, the traditional fourth-order Runge-Kutta approach necessitated double the effort of the modified Euler method and quadruple that of the Euler method. Although the Euler method exhibits significant inaccuracy, the revised Euler approach effectively mitigated it to a degree.

In the numerical solution of ordinary differential equations, there are two types of errors, which are round-off errors and truncation errors. Rounding errors arise from the fact that computers only characterize numbers with a fixed and limited number of significant figures; as a result, such numbers cannot be adequately represented in computer memory, while truncations arise when approximations are used to determine a number of quantities. A numerical method is said to be convergent if:

$$\lim_{h \rightarrow 0} \max_{1 \leq n \leq N} |y(x_n) - y_n| \quad (1)$$

where $y(x_n)$ represents the approximate solution, and y_n represent the exact solution [1].

The study focused on solving ordinary differential equations (ODEs) using analytical methods, graphical methods, and numerical methods. Numerical methods are typically employed when exact solutions cannot be obtained through familiar methods due to complex or impractical situations [8]. Therefore, this article applied different step sizes for each numerical method to evaluate their effectiveness. The results indicated that smaller step lengths were necessary to achieve more accurate outcomes. This underscores the importance of carefully selecting step sizes in numerical methods to obtain precise solutions for differential equations.

Next, since most ordinary differential equations cannot be solved analytically, numerical computations are the only way to learn about the solution. Various forms of ordinary differential equations were solved accurately and concluded that Runge-Kutta is a well-known and popular method as it approaches with the most efficiency [9]. The study also concludes that the fourth-stage Runge-Kutta method (RK4) is more effective and efficient in finding

approximate solutions of IVP than the Euler method and modified Euler method, as its numerical solution moves toward the analytical solution, and the error of RK4 is very less than other both methods [8].

Differential equations are widely utilized in science and engineering for mathematical modeling and have been used to reformulate mathematical issues such as ordinary differential equations and partial differential equations. As the differential equation that models the problem is too hard to solve precisely most of the time, the approach of using methods for approximating the solution of the original problem is applied [10].

3. Methodology

The formulas involved in this article are the Euler Method, Modified Euler Method, and Runge-Kutta Method. The Euler method, considered the most basic among the three methods, approximates the solution curve using tangents and estimates the next point at each step by utilizing the derivative at the current point. This method's formula is:

$$y_{n+1} = y_n + hf(x_n, y_n) \quad (2)$$

It offers the advantages of being simple to implement and computationally efficient, but suffers from poor accuracy, especially in the case of stiff equations, and tends to accumulate errors with larger step sizes.

The modified Euler method is an improvement over Euler's method; it provides a better slope estimation, and it computes two approximations and an average of them. The formula for the modified Euler method is:

$$y_{n+1} = y_n + hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1\right) \quad (3)$$

where,

$$k_1 = f(x_n, y_n), k_2 = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1\right) \quad (4)$$

$$y_{n+1} = y_n + hk_2 \quad (5)$$

The third method is the Runge-Kutta method, which is a family of numerical methods for solving ODEs. The most commonly used is the fourth-order Runge-Kutta method (RK4). It calculates the slope at multiple points within the interval and takes a weighted average of these slopes to approximate the next value. The formula for the Runge-Kutta method is:

$$y_{n+1} = y_n + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4) \quad (6)$$

where,

$$k_1 = hf(x_n, y_n) \quad (7)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \quad (8)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) \quad (9)$$

$$k_4 = hf(x_n + h, y_n + k_3) \quad (10)$$

$$k = \frac{(k_1 + 2k_2 + 2k_3 + k_4)}{6} \quad (11)$$

This article uses the three methods above to solve the three problems as below:

i) first order ODE:

$$\frac{dy}{dx} = x^2 - y; y(0) = 1. \quad (12)$$

ii) second order ODE:

$$\frac{d^2y}{dx^2} = 3 \frac{dy}{dx} - 2y; y(0) = 1; y'(0) = 0 \tag{13}$$

iii) third order ODE:

$$\frac{d^3y}{dx^3} + 2 \frac{d^2y}{dx^2} - 3 \frac{dy}{dx} + 2y = 0; y(0) = 1; y'(0) = 0; y''(0) = 2. \tag{14}$$

All the calculations were computed using MATLAB. The results were defined by comparing both the approximate solution to its exact solution and the approximate error of the numerical methods for each equation. This comparison helped determine which method achieved better accuracy.

4. Results and Discussion

Fig. 1 to Fig. 3 illustrate that all three methods display comparable errors for each solution during the initial iteration. However, as the iterations proceed, the Euler method consistently yields higher error values compared to the modified Euler and Runge-Kutta methods. Upon analyzing the error results for each equation, it becomes apparent that the modified Euler method consistently yields the lowest error values. This trend underscores the accuracy of the Modified Euler method compared to both the Euler and Runge-Kutta methods across various iterations.

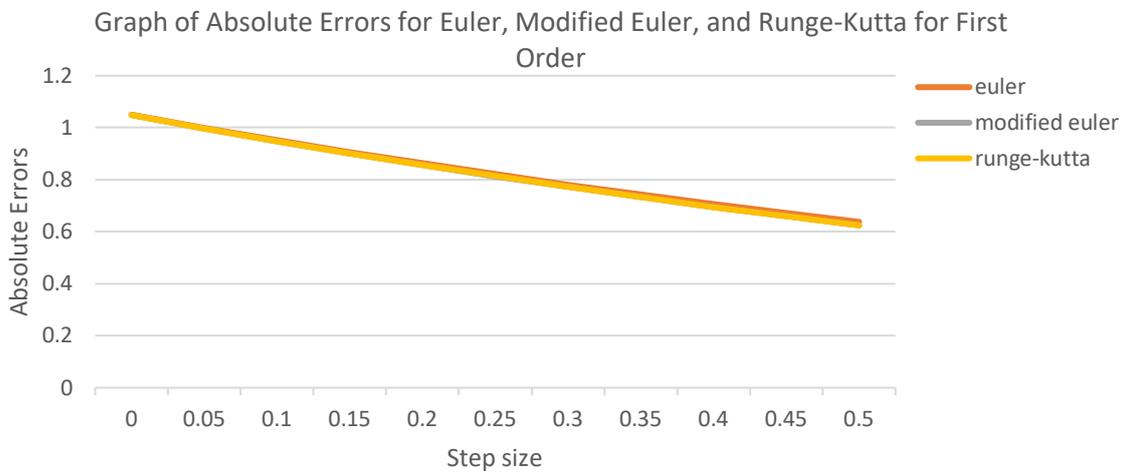


Fig. 1 Graph of Absolute Errors for Euler, Modified Euler, and Runge-Kutta methods for First Order ODE

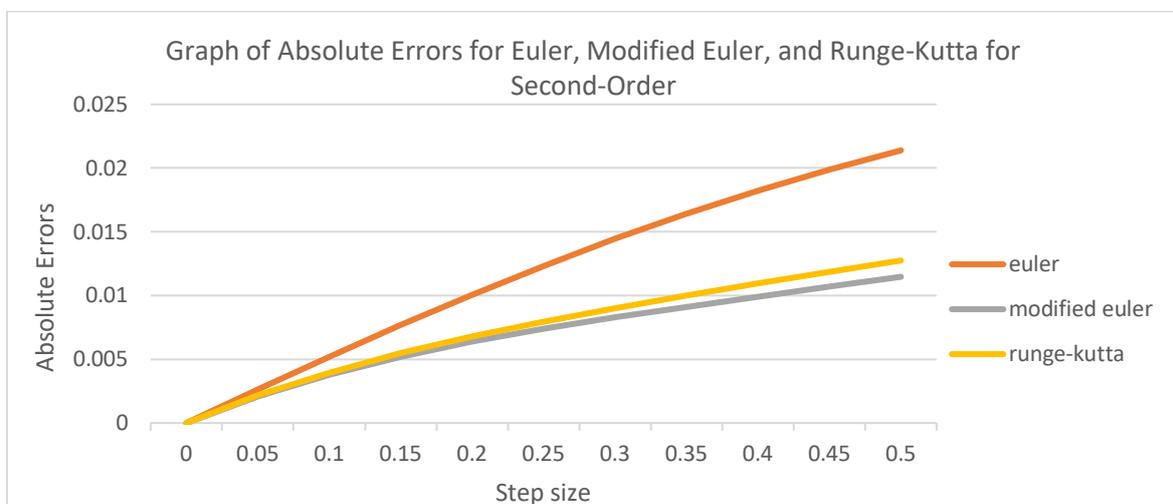


Fig. 2 Graph of Absolute Errors for Euler, Modified Euler, and Runge-Kutta methods for Second Order ODE

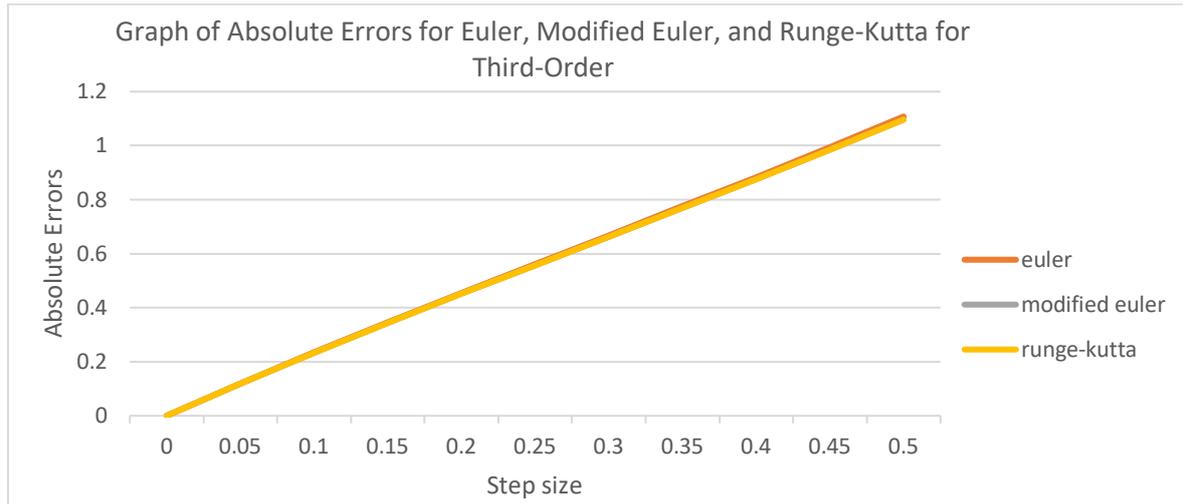


Fig. 3 Graph of Absolute Errors for Euler, Modified Euler, and Runge-Kutta methods for Third Order ODE

The results demonstrate that the Modified Euler Method consistently outperforms both Euler's and Runge-Kutta methods, particularly in terms of minimizing errors. This performance can be attributed to the nature of the Modified Euler Method, which improves on the basic Euler Method by using a two-step approximation. In the Modified Euler method, an initial prediction is made using the standard Euler method, and then this prediction is corrected by considering the slope at the midpoint of the interval. This additional correction step results in higher-order approximation, reducing the truncation error and providing a more accurate solution compared to the basic Euler method, which only uses the slope at the beginning of the interval. While the Runge-Kutta Method (RK4) is known for its high accuracy, the Modified Euler Method provides a comparable level of precision with less computational overhead. The Runge-Kutta method requires multiple intermediate steps within each iteration, leading to a computational cost, especially for larger systems or longer time intervals.

5. Conclusion

In summary, we have explored three effective methods for solving ordinary differential equations (ODEs) with initial value problems (IVPs). This study identifies the Modified Euler Method as optimal for lower-order ODEs, while Runge-Kutta excels for higher-order cases. The Modified Euler Method emerges as the optimal choice for solving first- and second-order ODEs, offering reliable accuracy. Conversely, for third-order equations, the Runge-Kutta method demonstrates superiority by consistently generating the smallest errors. However, among the three approaches, the Euler Method consistently yields the highest error values for ODEs of all orders. Thus, careful consideration of the specific characteristics of the ODE and desired accuracy level is essential in selecting the most appropriate numerical method.

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There is no finding to this article.

Conflict of Interest

Authors declare that there is no conflict of interests regarding the publication of the paper.

Author Contribution

The authors confirm contribution to the paper as follows: **study conception and design:** Izani Salleh, Amiruddin Ab Aziz; **data collection:** Izani Salleh; **analysis and interpretation of results:** Izani Salleh, Amiruddin Ab Aziz Z; **draft manuscript preparation:** Nur Afriza Baki. All authors reviewed the results and approved the final version of the manuscript.

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