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# Solving Third Order Ordinary Differential Equation (ODE) by using Differential Transform Method (DTM)

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**Abstract:** In this research, a semi analytical solution called differential transform method (DTM) was used to solve Third Order Ordinary Differential Equation (ODE). The Third Order ODE's approximative solutions were discovered using Maple 2020. This method has been shown to reduce computational work and save time. The numerical result derived by DTM is compared with the result of Bernstein Polynomials Method (BPM) [1], Adaptive Polynomial Method (APM) [2] and the Modified Adomian Decomposition Method (MADM) [3] which are based on earlier research in order to evaluate how closely the value matches the exact value. After comparing the results of the various approaches, it was found that DTM provided the closest approximation for solving Third Order ODEs. This outcome depends on the kind of Third Order ODE and the number of terms in the DTM. For solving Third Order ODE, a few terms of DTM are recommended to obtain the approximate value. DTM have successfully solved three example and regarded as the simplest method to use and may solve differential problems since it requires less computational to produce an approximation to exact value.

**Keywords**: Third Order Ordinary Differential Equation, Differential Transform Method,

## 1. Introduction

The Taylor series expansion is an analytical and computational methodology for solving problems involving differential equations in the form of polynomials that are used in the differential transform method (DTM). [4] was the first to introduce the differential transform, and its main application in

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electric circuit analysis is to deal with both linear and nonlinear starting value problems. In terms of known and unknown boundary conditions, the DTM quickly calculates the exact values of an analytical function's n-th derivative at a given place. For differential equations, this approach produces an analytical solution in the form of a polynomial. It varies from the standard high order Taylor series approach, which requires the symbolic computation of the required derivatives of the data functions. The Taylor series approach requires a long time to compute for large orders. The DTM is an iterative approach for obtaining analytical Taylor series solutions to differential equations by [5].

The search for general approaches to integrating differential equations began when [6] classified first order differential equations into three groups. The first two classes, which are now known as ordinary differential equations, only comprise ordinary derivatives of one or more dependent variables about a single independent variable. Partial derivatives of a single dependent variable, often known as partial differential equations, fell under the third category [7].

Several researchers have looked at and recommended the optimal technique for directly solving the ODE system. For solving generic third order ODEs, [8] proposed a P-stable linear multistep technique. A P-stable multistep approach for solving generic third order IVPs of ODEs directly with constant step size based on the collocation of the differential system from a basis function. For solving certain third order ODEs, [9] developed a direct six-step block strategy with a fixed step size. During our analysis, we have been selected to solve the third order ODEs with DTM to prove that DTM is much more precise and effective at solving third order ODEs.

For solving a large range of differential equations, this method is effective and precise. DTM has piqued the interest of many academics due to its benefits, and various attempts have been made by scholars to extend this technology to solve other sorts of equations. Published a study illustrating how to solve SIS and SI epidemic models using the differential transformation technique [10]. The DTM is used for a constant population to develop semi-analytical SIS and SI epidemic model solutions. The authors proved DTM's convergence analysis using two theorems. The exact solution is then compared to the numerical computation. This study demonstrates that raising the order of DTM improves the accuracy of the approximation solution, allowing for increased precision. This study will look at the DTM's efficiency in solving third order ordinary differential equations. The DTM will be used to solve various third order ordinary differential equation, with the results compared to the exact solution.

An ordinary differential equation (ODE) is a mathematical expression that consists of one or more functions of a single independent variable, as well as their derivatives. A differential equation has one or more derivatives and a function. The derivative of functions for a single independent variable is referred to as ordinary in ODE. Only one independent variable and one or more of its derivatives with respect to the variable are included in this equation by [11].

The general formula of Third Order ODE is

$$y''' = f(x, y, y', y'') \qquad \text{Eq. 1}$$

with initial condition

$$y(x_0) = a, y'(x_0) = b, y''(x_0) = c \qquad \text{Eq. 2}$$

where $a$, $b$ and $c$ are constant.

This research aims to analyse a solution of third order ordinary differential equation (ODE) with its initial condition by applying the differential transform method (DTM). According to previous research, the numerical approach is usually used to solve problems, and when the analytical method is used to solve the problem, it becomes more difficult to solve. In this research, we want to see what happen after applying the fundamental mathematical equation on the third order ODE then the outcome will compare to the exact solution of this issue as well as other methods. According to the general introduction, DTM is a semi-analytical technique therefore, in this study, we explored the semi-analytical method to see if it is effective. Many mathematicians regard this method as a quick convergence method that produces a series of replies that efficiently converge to an approximate solution. Various researchers realized that DTM delivers a series of solutions that efficiently converge to an approximate solution. Therefore, to get the objective such as to explore the method of differential transform method (DTM) for solving third order ordinary differential equation (ODE), to analyse third order ODE by DTM for finding an analytical solution and to develop an algorithm of DTM that follows the given third order ODE, three examples are used to demonstrate the DTM's efficiency and accuracy. The obtained results are then compared to the exact solution and several methods.

## 2. Methodology

The differential transform of the $y(x)$ function for the $k$th derivative is defined as follows if the function $y(x)$ is continuously differentiable

$$Y(k) = \frac{1}{k!}\left[\left.\frac{d^k y(x)}{dx^k}\right|_{x=0}\right]. \qquad \text{Eq. 3}$$

Where $Y(k)$ is the transformed function and $y(x)$ is the original function. $Y(k)$, the differential inverse transform is defined as

$$y(x) = \sum_{k=0}^{\infty} Y(k)(x - x_0)^k \qquad \text{Eq. 4}$$

The fundamental mathematical operations performed by differential transform are easily accessible and are given in Table 1:

**Table 1: The fundamental operation performed by differential transform method**

| Original functions | Transformed functions |
|---|---|
| $y(x) = g(x) \pm h(x)$ | $Y(k) = G(k) \pm H(k)$ |
| $y(x) = cg(x)$ | $Y(k) = cG(x)$, where $c$ is constant |
| $y(x) = y'(x)$ | $Y(k) = (k+1)Y(k+1)$ |
| $y(x) = y''(x)$ | $Y(k) = (k+1)(k+2)Y(k+2)$ |
| $y(x) = y^m(x)$ | $Y(k) = \frac{(k+m)!}{k!}Y(k+m)$ |
| $y(x) = \alpha x^m$ | $Y(k) = \alpha\delta(k-m); where\ \delta(k-m) = \begin{cases} 1 & \text{if } k = m \\ 0 & \text{if } k \neq m \end{cases}$ |
| $y(x) = e^{ay}$ | $Y(k) = \frac{\lambda^k}{k!}$ |

## 3. Result and Discussion

Example 1. We consider the following Third Order ODE as follows by [1].

$$y''' + 5y'' + 7y' + 3y = 0 \qquad \text{Eq. 5}$$

With initial condition

$$y(0) = 1, y'(0) = 0, y''(0) = -1 \qquad \text{Eq. 6}$$

And given the exact solution,

$$y(x) = e^{-x} + xe^{-x} \qquad \text{Eq. 7}$$

From the initials condition and theorem, we obtain

$$Y(0) = 1\,, Y(1) = 0\,, Y(2) = -\frac{1}{2} \qquad \text{Eq. 8}$$

By multiplying Eq. 5 by $x$ and using the theorem, the recurrence relation is obtained

$$Y(k+3) = \frac{k!}{(k+3)!}\left(-5\left(\frac{(k+2)!}{k!}Y(k+2)\right) - 7\big((k+1)Y(k+1)\big) - 3(Y(k))\right)$$

Eq. 9

Substitute *Eq.* 8 into *Eq.* 9 at $k = 0$, we have

$$Y(3) = \frac{1}{3}$$ *Eq.* 10

The recurrence relation Eq. 5 at $k = 1,2,3,4$, we obtain

$$Y(4) = -\frac{1}{8}$$
$$Y(5) = \frac{1}{30}$$
$$Y(6) = -\frac{1}{144}$$
$$Y(7) = \frac{1}{840}$$

Therefore, combine all the term from the term above and do the Taylor series until 7th term

$$Y(x) = 1 - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{8}x^4 + \frac{1}{30}x^5 - \frac{1}{144}x^6 + \frac{1}{840}x^7$$ Eq. 11

Figure 1 shows that the DTM are close with the exact solution same goes with Bernstein Polynomial Method (BPM) same as in [1], which the author got the solution of solving directly third order ODEs using operational matrices of Bernstein polynomials method with applications to fluid flow equations. Table 2 shows the result of DTM are very near with the exact solution. Table 3 shows that even through DTM solution are very near with exact solution in term of accuracy BPM are much better.

Example 2. Consider the following Third Order ODE as follows by [2].

$$y''' + 2y'' - y' - 2y = e^x$$ Eq. 12

With initial condition

$$y(0) = 1, y'(0) = 2, y''(0) = 0$$ Eq. 13

And given the exact solution,

$$y(x) = \frac{43}{36}e^x + \frac{1}{4}e^{-x} - \frac{4}{9}e^{-2x} + \frac{1}{6}xe^x$$ Eq. 14

From the initials condition and theorem, we obtain

$$Y(0) = 1, Y(1) = 2, Y(2) = 0$$ Eq. 15

By multiplying Eq. 12 by $x$ and using the theorem, the recurrence relation is obtained

$$Y(k+3) = \frac{k!}{(k+3)!}\left(-2\left(\frac{(k+2)!}{k!}Y(k+2)\right) + (k+1)Y(k+1) + 2Y(k) + \frac{1^k}{k!}\right)$$

Eq. 16

Substitute Eq. 15 into Eq. 16 at $k = 0$, we obtain

$$Y(3) = \frac{5}{6}$$ Eq. 17

The recurrence relation Eq. 12 at $k = 1,2,3,4$, we obtain

$$Y(4) = -\frac{5}{24}$$
$$Y(5) = \frac{2}{15}$$
$$Y(6) = -\frac{13}{360}$$
$$Y(7) = \frac{59}{5040}$$

Therefore, combine all the term from the previous term and do the Taylor series until 7th term.

$$Y(x) = 1 + 2x + \frac{5}{6}x^3 - \frac{5}{24}x^4 + \frac{2}{15}x^5 - \frac{13}{360}x^6 + \frac{59}{5040}x^7 \quad \text{Eq. 18}$$

Figure 2 depicts the trend of DTM solution is very near with exact solution same goes with the Adaptive Polynomial Method (APM) solution like [2], which the author got the solution of Adaptive Polynomial Method for Solving Third-Order ODE With Application in Thin Film Flow. The table 4 shows the numerical result of DTM. In table 5, the absolute error of DTM is closer with the exact solution but APM closer with the exact solution. This result shows that the APM is better than DTM.

Example 3. Consider the following Third Order ODE as follows by [3].

$$y''' + 4y' = x \quad \text{Eq. 19}$$

With initial condition

$$y(0) = 0, y'(0) = 0, y''(0) = 1 \quad \text{Eq. 20}$$

And given the exact solution,

$$y(x) = \frac{3}{16}(1 - \cos 2x) + \frac{1}{8}x^2 \quad \text{Eq. 21}$$

From the initials condition and theorem, we obtain

$$Y(0) = 0, Y(1) = 0, Y(2) = \frac{1}{2} \quad \text{Eq. 22}$$

By multiplying Eq. 19 by $x$ and using the theorem, the recurrence relation is obtained

$$Y(k+3) = \frac{k!}{(k+3)!}\big(\delta(k-1) - 4(k+1)Y(k+1)\big) \quad \text{Eq. 23}$$

Substitute Eq. 22 into Eq. 23 at $k = 0$, we obtain

$$Y(3) = 0 \quad \text{Eq. 24}$$

The recurrence relation Eq. 19 at $k = 1,2,3,4$, we obtain

$$Y(4) = -\frac{1}{8}$$
$$Y(5) = 0$$
$$Y(6) = \frac{1}{60}$$
$$Y(7) = 0$$

Therefore, combine all the term from the previous term and do the Taylor series until 7th term

$$Y(x) = \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{60}x^6 \quad \text{Eq. 25}$$

Figure 3 shows the DTM was compared with the Modified Adomian Decomposition Method (MADM) and exact solution. [3] the author got the solution of The Modified Adomian Decomposition Method for the Solution of Third Order Ordinary Differential Equations. Table 6 shows the closeness of DTM with exact solution are based on the smallest absolute error. Table 7 shows that DTM much more accurate compared to the MADM.

**Table 2: Numerical solution of DTM for example 1**

| x | Exact Solution | DTM | Absolute error |
|---|---|---|---|
| 0 | 1.0000000000 | 1.0000000000 | 0.0000000000 |
| 0.1 | 0.9953211598 | 0.9953211598 | 0.0000000000 |
| 0.2 | 0.9824769037 | 0.9824769042 | 0.0000000005 |
| 0.3 | 0.9630636869 | 0.9630636979 | 0.0000000110 |
| 0.4 | 0.9384480644 | 0.9384481727 | 0.0000001083 |
| 0.5 | 0.9097959895 | 0.9097966271 | 0.0000006376 |
| 0.6 | 0.8780986178 | 0.8781013257 | 0.0000027079 |
| 0.7 | 0.8441950165 | 0.8442042005 | 0.0000091840 |
| 0.8 | 0.8087921354 | 0.8088185500 | 0.0000264146 |
| 0.9 | 0.7724823534 | 0.7725493386 | 0.0000669852 |
| 1.0 | 0.7357588824 | 0.7359126984 | 0.0001538160 |

**Table 3: Absolute error of the method used and minimum absolute error to exact solution for example 1**

| x | DTM | Bernstein Polynomial Method [1] |
|---|---|---|
| 0 | 0.0000000000 | 0.0000000000 |
| 0.1 | 0.0000000000 | 0.0000000000 |
| 0.2 | 0.0000000005 | 0.0000000000 |
| 0.3 | 0.0000000110 | 0.0000000000 |
| 0.4 | 0.0000001083 | 0.0000000000 |
| 0.5 | 0.0000006376 | 0.0000000001 |
| 0.6 | 0.0000027079 | 0.0000000001 |
| 0.7 | 0.0000091840 | 0.0000000001 |
| 0.8 | 0.0000264146 | 0.0000000001 |
| 0.9 | 0.0000669852 | 0.0000000001 |
| 1.0 | 0.0001538160 | 0.0000000001 |

**Table 4: Numerical solution of DTM for example 2**

| x | Exact Solution | DTM | Absolute error |
|---|---|---|---|
| 0 | 1.000000000 | 1.000000000 | 0.000000000 |
| 0.1 | 1.200813798 | 1.200813798 | 0.000000000 |
| 0.2 | 1.406373832 | 1.406373840 | 0.000000008 |
| 0.3 | 1.621112566 | 1.621112735 | 0.000000169 |
| 0.4 | 1.849234952 | 1.849236602 | 0.000001650 |
| 0.5 | 2.094830093 | 2.094839721 | 0.000009628 |
| 0.6 | 2.361970373 | 2.362010903 | 0.000040530 |
| 0.7 | 2.654801224 | 2.654937465 | 0.000136241 |
| 0.8 | 2.977624242 | 2.978012689 | 0.000388447 |
| 0.9 | 3.334975980 | 3.335952686 | 0.000976706 |
| 1.0 | 3.731704444 | 3.733928571 | 0.002224127 |

**Table 5: Absolute error of the method used and minimum absolute error to exact solution for example 2**

| x | DTM | Adaptive Polynomial Method [2] |
|---|---|---|
| 0 | 0.000000000 | 0.000000000 |
| 0.1 | 0.000000000 | 0.000000061 |
| 0.2 | 0.000000008 | 0.000000779 |
| 0.3 | 0.000000169 | 0.000002783 |
| 0.4 | 0.000001650 | 0.000005982 |
| 0.5 | 0.000009628 | 0.000009751 |
| 0.6 | 0.000040530 | 0.000013211 |
| 0.7 | 0.000136241 | 0.000015243 |
| 0.8 | 0.000388447 | 0.000014261 |
| 0.9 | 0.000976706 | 0.000008215 |
| 1.0 | 0.002224127 | 0.000004444 |

**Table 6: Numerical solution of DTM for example 3**

| x | Exact Solution | DTM | Absolute error |
|---|---|---|---|
| 0 | 0.0000000000 | 0.0000000000 | 0.000000000 |
| 0.1 | 0.4987516700 | 0.4987516667 | 0.00000000003 |
| 0.2 | 0.1980106360 | 0.1980106667 | 0.0000000030 |
| 0.3 | 0.4399957220 | 0.4399965000 | 0.0000000778 |
| 0.4 | 0.7686749200 | 0.7686826667 | 0.0000007746 |
| 0.5 | 0.1174433176 | 0.1174479167 | 0.0000045991 |
| 0.6 | 0.1645579210 | 0.1645776000 | 0.0000196790 |
| 0.7 | 0.2168811607 | 0.2169483167 | 0.0000671560 |
| 0.8 | 0.2729749104 | 0.2731690667 | 0.0001941563 |
| 0.9 | 0.3313503928 | 0.3318448500 | 0.0004944572 |
| 1.0 | 0.3905275318 | 0.3916666667 | 0.0011391349 |

**Table 7: Absolute error of the method used and minimum absolute error to exact solution for example 3**

| x | DTM | MADM [3] |
|---|---|---|
| 0 | 0.0000000000 | 0.000000000 |
| 0.1 | 0.00000000003 | 0.0000208055 |
| 0.2 | 0.0000000030 | 0.0003315586 |
| 0.3 | 0.0000000778 | 0.0016673278 |
| 0.4 | 0.0000007746 | 0.0052203302 |
| 0.5 | 0.0000045991 | 0.0125914046 |
| 0.6 | 0.0000196790 | 0.0257236790 |
| 0.7 | 0.0000671560 | 0.0468199615 |
| 0.8 | 0.0001941563 | 0.0782457118 |
| 0.9 | 0.0004944572 | 0.1224197072 |
| 1.0 | 0.0011391349 | 0.1816946904 |

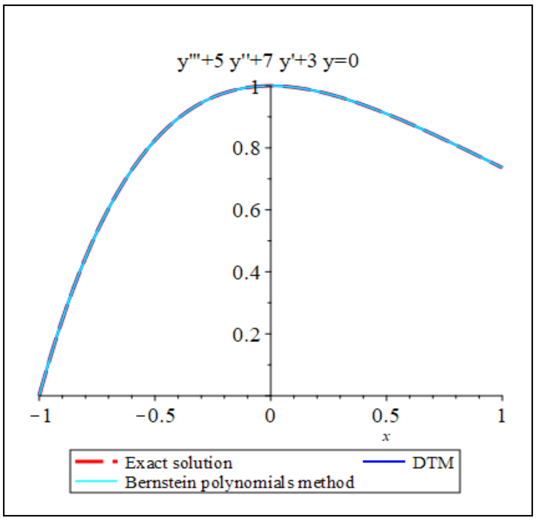


**Figure 1: The exact solution, DTM and Bernstein Polynomials Method of example 1**

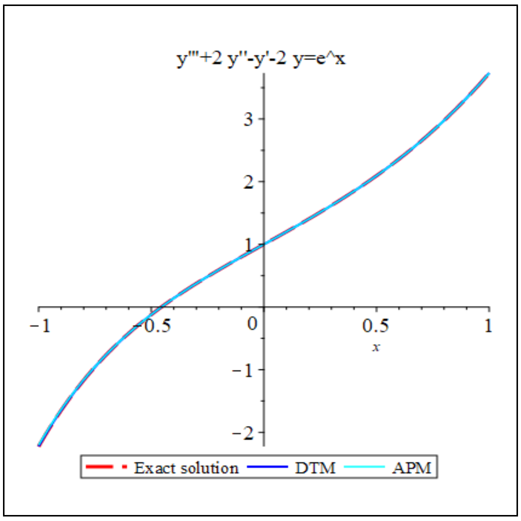


**Figure 2: The exact solution, DTM and Adaptive Polynomial Method (APM) of example 2**

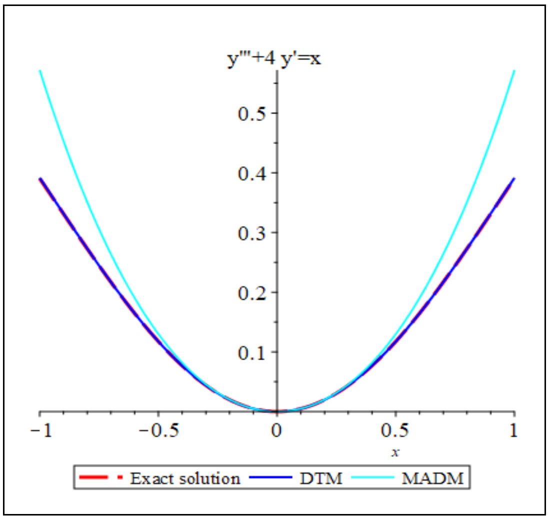


**Figure 3: The exact solution, DTM and The Modified Adomian Decomposition Method (MADM) of example 3**

## 4. Conclusion

In conclusion, higher term of DTM can improve the accuracy to the exact solution but higher term required more computational work for solving ODE. A few terms of DTM are recommended to obtain the approximate value. DTM have successfully solved the three example which is with BPM, APM and MADM. Due to less computational work to obtain an approximation to exact value, DTM is considered to be the simplest method to apply but yet can get closer to exact solution and can be solve differential problem. There are some recommendations that can be made for further study such as improve the algorithm of differential transform method (DTM) to obtain exact solution and use modified differential transform method with fractional newton by [12] or Adomian polynomials or Laplace transform or Pade approximation by [13] to improve the result to be close with the exact solution.

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