Enhanced Knowledge in Sciences and Technology Vol. 3 No. 2 (2023) 100-108
© Universiti Tun Hussein Onn Malaysia Publisher's Office

EKST

Homepage: http://publisher.uthm.edu.my/periodicals/index.php/ekst
e-ISSN : 2773-6385

Solving Fourth Order Ordinary Differential Equation by Using Differential Transform Method

Nur Fatin Kirana Hisamudin[1], Noor Azliza Abd Latif[1*]

[1] Department of Mathematics and Statistics,
Faculty of Applied Sciences and Technology,
Universiti Tun Hussein Onn Malaysia (Pagoh Campus),
84600 Pagoh, Muar, Johor, MALAYSIA

*Corresponding Author Designation

DOI: https://doi.org/10.30880/ekst. 2023.03.02.012
Received 15 January 2023; Accepted 3 February 2023; Available online 30 November 2023

Abstract: In this study, fourth order ordinary differential equation is solved by using a semi analytical solution which is differential transform method (DTM). A computational works done by Maple 2022 to obtain the approximate solutions of fourth order ordinary differential equation and proved that it saves amount of time and less work need to do. Numerical results obtained using DTM is compared with first block method (FBM) and Adomian decomposition method (ADM) by previous researchers to discover the accuracy of different method with exact solutions. The result of accuracy of DTM shows that it is depends on equation of fourth order ordinary differential equation and terms in DTM. DTM have succeed in solving three examples. Furthermore, DTM is considered as one of the simplest methods to apply in solving ordinary differential equation problems.

Keywords: Fourth Order Ordinary Differential Equation, Differential Transform Method

1. Introduction

Zhou [1] proposed the differential transform method (DTM) as an algebraic method for solving nonlinear and linear ordinary differential equation (ODE) [2]. Since the traditional method, Taylor series is too complex to solve differential equations, DTM offers structured and high capability method to solve both linear and nonlinear values [3], and also act as an extended version of Taylor series method [4]. Ayaz had successfully evolved a differential equation system by using DTM [5] and it is proven that DTM as an alternative method provides more benefits and approximate solutions [6].

Real-world problems such as in an engineering sector use ODE to apply theory and identify the best method of solution and real applications. The high ODE appears in various fields around the world. For example, fourth order ODE that controls the analysis of vibration of beam structure is connected with multi-boundary conditions which make it well solved by using high-order ODE. In this project, we want

*Corresponding author: azliza@uthm.edu.my

to explore on the method of DTM for solving fourth order ODE and analyse the analytical solution to develop an algorithm of DTM.

Fourth order ODE general form is

$$y'''' = f(x, y, y', y'', y'''), \quad \text{Eq. 1}$$

$$y(a) = \eta_0, y'(a) = \eta_1, y''(a) = \eta_2, y'''(a) = \eta_3. \quad \text{Eq. 2}$$

Many authors extensively studied about high order differential equations and provide various techniques to solve them. Many authors discovered the oscillation behaviour of high order ODE [7] and researched more about it [8] for higher order with continuously distributed delay [9].

The aim of this paper is to obtain fourth order ODE using DTM. Three examples are used to demonstrate the accuracy and precision of the DTM. The result obtained are then compared with the exact solution and two other methods.

2. Materials and Methods

The *k*th derivative of the differential transformation for the function *y(x)* is defined as follows:

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=x_0} \quad \text{Eq. 3}$$

where *y*(x) and *Y*(k) are the original function and the transform function respectively, and the differential inverse transformation of *Y*(k) is defined as

$$y(x) = \sum_{k=0}^{\infty} Y(k)(x - x_0)^k \; . \quad \text{Eq. 4}$$

The fundamental operations performed by differential transformation as follow in Table 1.

Table 1: The fundamental operation performed by differential transformation

Original functions	Transformed functions
$y(x) = g(x) \pm h(x)$	$Y(k) = G(k) \pm H(k)$
$y(x) = cg(x)$	$Y(k) = cG(x)$, where c is constant
$y(x) = y'(x)$	$Y(k) = (k+1)Y(k+1)$
$y(x) = y''(x)$	$Y(k) = (k+1)(k+2)Y(k+2)$
$y(x) = x^m$	$Y(k) = \delta(k-m) = \begin{cases} 1 & \text{if } k = m \\ 0 & \text{if } k \neq m \end{cases}$
$y(x) = y^m(x)$	$Y(k) = (k+1)(k+2)...(k+m)Y(k+m)$

$y(x)=e^{ay}$	$Y(k)=\begin{cases} e^{aY(0)} & \text{if } k=0 \\ a\sum_{r=0}^{k-1}\frac{r+1}{k}Y(r+1)F(k-r-1) & \text{if } k\geq 1 \end{cases}$

3. Results and Discussion

Example 1. Given fourth order ODE [10]

$$y'''' = x \quad \text{Eq. 5}$$

with initial conditions

$$y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = 0\ , \quad \text{Eq. 6}$$

where the exact solution given

$$y(x) = \frac{x^5}{120} + x \quad \text{Eq. 7}$$

By applying DTM, we will obtain as follows:

$$\frac{(k+4)!}{k!}Y(k+4) = \delta(k-1). \quad \text{Eq. 8}$$

By applying Table 1, the initial conditions can be transformed as

$$Y(0) = 0, Y(1) = 1, Y(2) = 0, Y(3) = 0 \quad \text{Eq. 9}$$

Then, at $k = 0,1,2,3$, we obtain

$$Y(4) = 0$$
$$Y(5) = \frac{1}{120}$$
$$Y(6) = 0$$
$$Y(7) = 0 \quad \text{Eq. 10}$$

Then, we combine all the terms to become a Taylor series as follow:

$$Y(x) = \frac{1}{120}x^5 + x + \cdots \quad \text{Eq. 11}$$

Figure 1 shows that the DTM is an effective method. It is because the data is aligned with the exact solution. Table 2 shows the result of DTM accurate with exact solution while Table 3 shows first block method solution also nearly accurate as exact solution which conclude the accuracy of DTM and first block method (FBM) for this example.

Table 2: Numerical solution of DTM for example 1

x	Exact Solution	DTM	Absolute error
0	0.0000000000	0.0000000000	0.0000000000
0.1	0.1000000833	0.1000000833	0.0000000000
0.2	0.2000026667	0.2000026667	0.0000000000
0.3	0.3000202500	0.3000202500	0.0000000000
0.4	0.4000853333	0.4000853333	0.0000000000
0.5	0.5002604167	0.5002604167	0.0000000000
0.6	0.6006480000	0.6006480000	0.0000000000
0.7	0.7014005833	0.7014005833	0.0000000000
0.8	0.8027306667	0.8027306667	0.0000000000

0.9	0.9049207500	0.9049207500	0.0000000000
1	1.0083333333	1.0083333333	0.0000000000

Table 3: Absolute error of DTM and FBM with minimum absolute error to exact solution for example 1

x	DTM	FBM	Minimum
0	0.0000000000	0.0000000000	0.0000000000
0.1	0.0000000000	0.0000000000	0.0000000000
0.2	0.0000000000	0.0000000000	0.0000000000
0.3	0.0000000000	0.0000000000	0.0000000000
0.4	0.0000000000	0.0000000000	0.0000000000
0.5	0.0000000000	0.0000000000	0.0000000000
0.6	0.0000000000	0.0000000000	0.0000000000
0.7	0.0000000000	0.0000000000	0.0000000000
0.8	0.0000000000	0.0000000000	0.0000000000
0.9	0.0000000000	0.0000000000	0.0000000000
1	0.0000000000	0.0000000000	0.0000000000

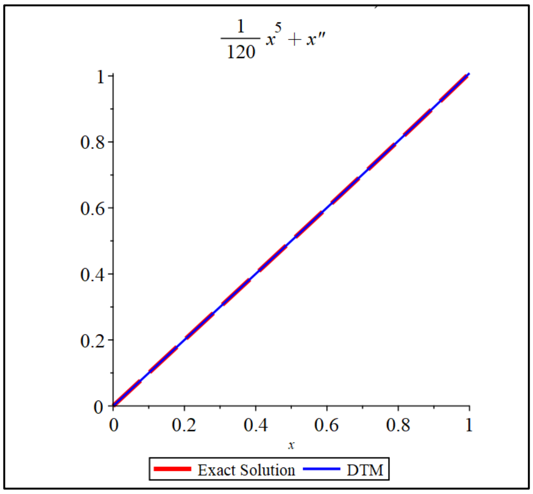

Figure 1: Exact solution and DTM

Example 2 for fourth order ODE is as follows [10]:

$$y'''' - 10y'' + 9y = 0 \quad \text{Eq. 12}$$

with initial conditions

$$y(0) = 5, y'(0) = -1, y''(0) = 21, y'''(0) = -49 \quad \text{Eq. 13}$$

where the exact solution is

$$y(x) = 4e^{x} - e^{-x} + 2e^{-3x} \quad \text{Eq. 14}$$

After applying DTM, we obtained

$$\frac{(k+4)!}{k!}Y(k+4) - 10\left[\frac{(k+2)!}{k!}Y(k+2)\right] + 9\left[\frac{(k+1)!}{k!}Y(k+1)\right] = 0 \quad \text{Eq.15}$$

The initial condition can be transformed as

$$Y(0) = 5, Y(1) = -1, Y(2) = \frac{21}{2}, Y(3) = \frac{-49}{6}$$

Eq. 16

Then, at $k = 0,1,2,3$, we obtain

$$Y(4) = \frac{73}{8}$$
$$Y(5) = -\frac{679}{120}$$
$$Y(6) = \frac{877}{240}$$
$$Y(7) = -\frac{8761}{5040}$$

Eq. 17

Then, we combine all the terms to become a Taylor series as follow:

$$Y(x) = 5 - x + \frac{21}{2}x^2 - \frac{49}{6}x^3 + \frac{73}{8}x^4 - \frac{679}{120}x^5 + \frac{877}{240}x^6 - \frac{8761}{5040}x^7 + \cdots$$

Eq. 18

Figure 2 shows that ADM is much closer with the exact solution compared to DTM and it proved that ADM is more accurate. Table 4 shows that ADM numerical solution is near to exact solution while Table 5 shows the DTM solution results.

Table 4: Numerical solution of DTM for example 2

x	Exact Solution	DTM	Absolute error
0	5.000000000	5.000000000	0.000000000
0.1	4.997482695	4.997692730	0.000021003
0.2	5.164503551	5.167667617	0.000316406
0.3	5.471756330	5.486946473	0.015190143
0.4	5.899367170	5.945111447	0.045744277
0.5	6.434614744	6.541172185	0.106557441
0.6	7.070261340	7.280435703	0.210174363
0.7	7.803338381	8.170502847	0.367164466
0.8	8.634270655	9.215515269	0.581244614
0.9	9.566253810	10.40777678	0.841522970
1	10.60482201	11.71587301	1.111051000

Table 5: Absolute error of DTM and ADM with minimum absolute error to exact solution for example 2

x	DTM	ADM	Minimum
0	0.000000000	0.000000000	0.000000000
0.1	0.000021003	0.000002198	0.000002198
0.2	0.000316406	0.000135231	0.000135231
0.3	0.015190143	0.001481306	0.001481306
0.4	0.045744277	0.008011010	0.008011010

0.5	0.106557441	0.000000000	0.000000000
0.6	0.210174363	0.029438962	0.029438962
0.7	0.367164466	0.084747100	0.367164466
0.8	0.581244614	0.206174992	0.206174992
0.9	0.841522970	0.443517375	0.443517375
1	1.111051000	0.868599343	0.868599343

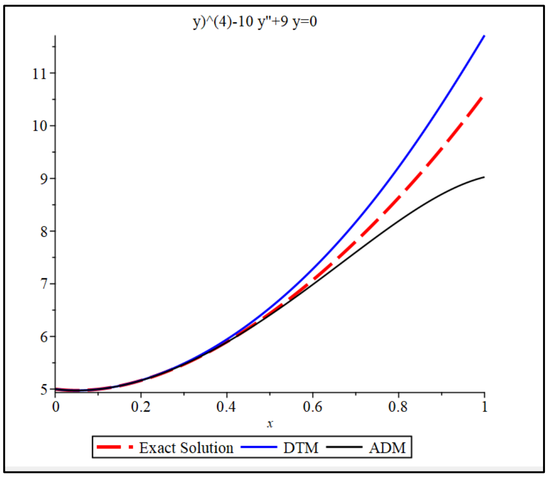

Figure 2: Exact solution, DTM and ADM solution.

Example 3. Considered as follow [10]:

$$y^4 - 3y'' - 4y = 0 \quad \text{Eq. 19}$$

with initial condition

$$y(0) = 1, y'(0) = \frac{1}{3}, y''(0) = 0, y'''(0) = 0 \quad \text{Eq. 20}$$

and its exact solution is

$$y(x) = \frac{1}{3}\left[\frac{7}{20}e^{2x} + \frac{1}{4}e^{-x} + \frac{12}{5}cosx + \frac{4}{5}sinx\right] \quad \text{Eq. 21}$$

After applying DTM, we obtained

$$\frac{(k+4)!}{k!}Y(k+4) - 3\left[\frac{(k+2)!}{k!}y(k+2)\right] - 4\left[\frac{(k+1)!}{k!}Y(k+1)\right] = 0 \quad \text{Eq. 22}$$

Besides, the initial condition can be transformed into

$$Y(0) = 1, Y(1) = \frac{1}{3}, Y(2) = 0, Y(3) = 0 \quad \text{Eq. 23}$$

Then, at $k = 0,1,2,3$, we obtain

$$Y(4) = \frac{1}{18}$$
$$Y(5) = 0$$
$$Y(6) = \frac{1}{180}$$
$$Y(7) = \frac{1}{945} \quad \text{Eq. 24}$$

Then, we combine all the terms to become a Taylor series as follow:

$$Y(x) = 1 + \frac{1}{3}\, x \, + \frac{1}{18}x^4 + \frac{1}{180}x^6 + \frac{1}{945}x^7 + \cdots$$

Eq. 25

Figure 3 compares the result of numerical solution, DTM and ADM which shows that ADM is more accurate compare to DTM which has slightly difference with exact solution. DTM method is convert from 0 until 0.4 and started to divert from 0.5 until infinity. To avoid this matter from continue divergent, DTM method should be modified. This conclude that ADM is better than DTM.

Table 6: Numerical solution of DTM for example 3

x	Exact Solution	DTM	Absolute error
0	1.000000000	1.000000000	0.000000000
0.1	1.040525683	1.033338895	0.0007186788
0.2	1.079305528	1.066755926	0.0125496020
0.3	1.117389958	1.100454281	0.0169356770
0.4	1.156200132	1.134780045	0.0214200870
0.5	1.197589961	1.170233962	0.0273559990
0.6	1.243921095	1.207488823	0.0003491880
0.7	1.298153904	1.247412975	0.0507409290
0.8	1.363958188	1.291100500	0.0728576880
0.9	1.445848160	1.339908584	0.1059395760
1	1.549347273	1.395502646	0.1538446270

Table 7: Absolute error of DTM and ADM with minimum absolute error to exact solution for example 3

x	DTM	ADM	Minimum
0	1.000000000	0.000000000	0.000000000
0.1	1.033338895	0.000717555	0.000717555
0.2	1.100454281	0.012367560	0.012367560
0.3	1.134780045	0.016000608	0.016000608
0.4	1.170233962	0.018416527	0.018416527
0.5	1.207488823	0.019891238	0.019891238
0.6	1.247412975	0.020650613	0.020650613
0.7	1.291100500	0.020887406	0.020887406
0.8	1.339908584	0.020781881	0.020781881
0.9	1.395502646	0.020529389	0.020529389
1	1.100454281	0.020379019	0.020379019

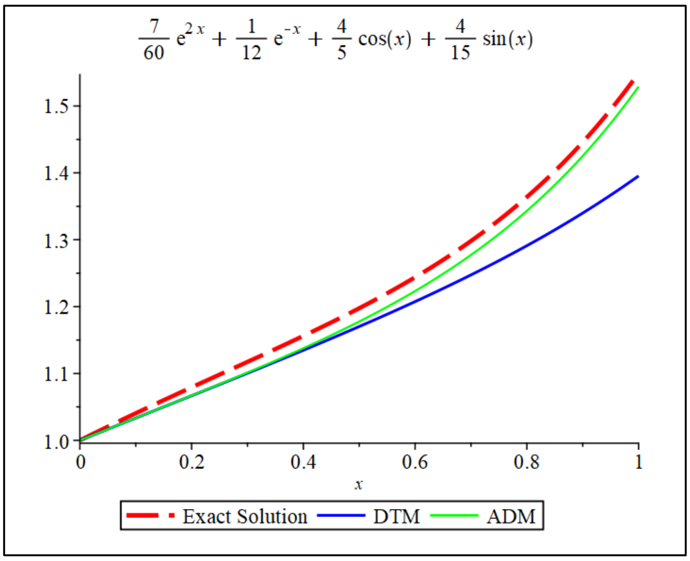

Figure 3: Exact solution, DTM and ADM solution

4. Conclusion

DTM can be improved by modifying it to become into a higher term which will provide more accurate result to exact solution but it requires more complex computational work. In this project, DTM has successfully solve the three examples of fourth order ordinary differential equation. Although some results might differ than exact solution, but it still follows the trend and considered as less computational and easy way to solve ODE problem by using Maple 2022. Improvements can be made by two recommendations which are improve DTM algorithm to make it more accurate to exact solution and modify the differential transform method.

Acknowledgement

The authors would also like to thank the Faculty of Applied Sciences and Technology, Universiti Tun Hussein Onn Malaysia for its support.

References

[1] Lien, C.H & Prudhomme, S. (2012). ISRN Applied Mathematics. Extension of Zhou's Method to Neutral Functional-Differential Equation with Proportional Delays, 518361, 1-10.

[2] Odibat, Z. M. (2008). Mathematical and Computer Modelling. Differential transform method for solving Volterra integral equation with separable kernels, 48(7-8), 1144-1149.

[3] Sepasgozar, S., Faraji, M., & Valipour, P. (2017). Propulsion and Power Research. Application of differential transformation method (DTM) for heat and mass transfer in a porous channel, 6(1), 41-48.

[4] Jang, M. J., & Chen, C. L. (1998). Applied Mathematics and Computation. Analysis of the response of a strongly nonlinear damped system using a differential transformation technique, 88(2-3), 137-151.

[5] Ayaz, F. (2004). Applied Mathematics and Computation. Applications of differential transform method to differential-algebraic equations, 152(3), 549-657.

[6] Hatami, M., & Jing, D. (2016). Alexandria Engineering Journal. Differential transformation method for Newtonian and non-Newtonian nanofluids flow analysis: Compared to Numerical Solution, 55(2), 731-739.

[7] Zhang, C., Agarwal, R. P., Bohner, M. & Li, T. (2013). Applied Mathematics Letter. New results for oscillatory behaviour of even-order half-linear delay differential equations, 26(2), 179-183.

[8] Bazighifan, O. (2017). Oscillatory behaviour of higher-order delay di Erential equations, 2(3), 105-110.

[9] Bazighifan, O., Elabbasy, E. M. & Moaaz, O. (2017). Oscillation Criteria for Fourth-Order Nonlinear Differential, 15, 50–5.

[10] Singh, D. I., & Singh, G. (2019). International Journal of Research in Engineering Application & Management. Numerical Study for solving fourth order ordinary differential equations, 5(1).