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# Mathematical Modelling of Inviscid Fluid Flows in Thin Stenosed Elastic Tube Using Reductive Perturbation Method

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**Abstract**: Mathematical modelling of the wave propagation of blood flow gives useful information to medicine. This paper presents an analytical study on the wave propagation of blood flow in the stenosed artery. First, the artery is treated as a thin-walled prestressed elastic tube with stenosis. By considering blood as an incompressible inviscid fluid, a mathematical model of nonlinear wave propagation in the thin-walled elastic tube with stenosis is proposed. Then, by applying the reductive perturbation method to the nondimensional equations of tube and fluid, a set of various orders of differential equations is obtained. As a result, the partial differential equation for the incompressible inviscid fluid flow in the stenosed tube is proved to be the Korteweg-de Vries (KdV) equation with variable coefficients, and its analytical solution is determined. From graphical outputs, the fluid passes through the stenosis, and the amplitude of the wave and fluid pressure decreases as time increases, but the amplitude of the wave and fluid pressure increases as time increases after the fluid passes through the stenosis. In addition to this, when the height of stenosis is higher, the wave speed is lower. Finally, the increase in circumferential stretch causes the wave speed to decrease, and the peak-to-peak of the wave also becomes wider.

**Keywords**: Inviscid Fluid, Thin-Walled Elastic Tube, Stenosis, Reductive Perturbation Method, Korteweg-de Vries Equation

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## 1. Introduction

A blood circulatory system involves organs such as the heart and the whole network of blood vessels in the human body. There are two circuits, which are the pulmonary system and the systemic system, and both circuits start and finish in the heart [1].

Thomas Young made important contributions to hemodynamics. He calculated the pressure wave speed in an incompressible liquid filled in an elastic tube. Unfortunately, Young's analysis was cryptic, and the wave speed was not explicitly stated, so his work was undiscovered until the German brothers Weber unearthed it almost half a century later [2]. In 2007, Demiray studied wave propagation in fluid-filled elastic tubes with stenosis. In his study, the arteries are treated as thin-walled prestressed elastic tubes with stenosis, whereas the blood is an inviscid fluid. Based on the balance between nonlinearity and dispersion, the variable coefficients KdV and modified KdV equations are acquired by using the reductive perturbation method. The results in the study showed that wave speeds are maximum at the centre of stenosis and the values decreases when it goes away from the stenosis.

Yang, Song and Yang [3], investigated the propagation of pulse wave in a deformable artery filled with inviscid blood using the reductive perturbation technique and found that the higher order nonlinear and dispersion terms lead to the distortion of the wave, while the initial deformation of the tube wall will influence the wave amplitude and wave width. Dhange, Sankad and Bhujakkanavar [4] investigated the model of blood flow with stenosis and discovered that arterial stenosis lowers arterial blood flow.

In recent years, there is a growing interest in studying the nonlinear wave propagation of blood flow in the artery. Mostly, the research primarily focused on the wave propagation in the artery without the stenosis. In the human body, there exists a phenomenon of narrowing arteries, which can be due to lipid deposits or plaque in arteries, which affects the blood flow throughout the body. As a result, the artery is treated as an incompressible, inhomogeneous, isotropic, and pre-stressed uniform thin elastic tube with stenosis in it, and the blood is treated as an incompressible inviscid fluid.

Therefore, in our study, the reductive perturbation method is applied to investigate the propagation of solitary waves in incompressible, inhomogeneous, isotropic, and pre-stressed thin elastic tube with stenosis filled with inviscid fluid. With this aim, three objectives of the study are planned. The first objective is to derive the nonlinear partial differential equation of wave propagation in inviscid fluid filled in thin elastic tube with stenosis. The second objective is to obtain the progressive wave solution for nonlinear partial differential equation. The third objective is to analyse the solution of progressive wave on the variation of radial displacement, wave speed and fluid pressure. The progressive wave solutions are then presented graphically using MATLAB and are discussed.

## 2. Basic Equations

In this section, some basic equations for wave propagation of blood flow are given.

### 2.1 Equation of Tube

The artery is identified as the thin elastic tube with stenosis. The model for the incompressible, inhomogeneous, isotropic, and pre-stressed thin elastic tube with stenosis is shown in Figure 1, $r_0$ is the deformed radius at the coordinate system's origin, $Z^*$ is the axial coordinate before deformation, the axial coordinate after static deformation is denoted as $z^*$, $f(z^*)$ is a function which characterizes the axially symmetric stenosis on the surface of the arterial wall and $u^*$ is the dynamical radial displacement.

The equation motion of tube in the radial direction [5] is expressed as

$$-\frac{\mu}{\lambda_z}\frac{\partial \Sigma}{\partial \lambda_2}+\mu R_0\frac{\partial}{\partial z^*}\left\{\frac{\left(-f^{*\prime}+\frac{\partial u^*}{\partial z^*}\right)}{\left[1+\left(-f^{*\prime}+\frac{\partial u^*}{\partial z^*}\right)^2\right]^{\frac{1}{2}}}\frac{\partial \Sigma}{\partial \lambda_1}\right\}+\frac{P^*}{H}(r_0-f^*+u^*)=\rho_0\frac{R_0}{\lambda_z}\frac{\partial^2 u^*}{\partial t^{*2}} \qquad Eq.\ 1$$

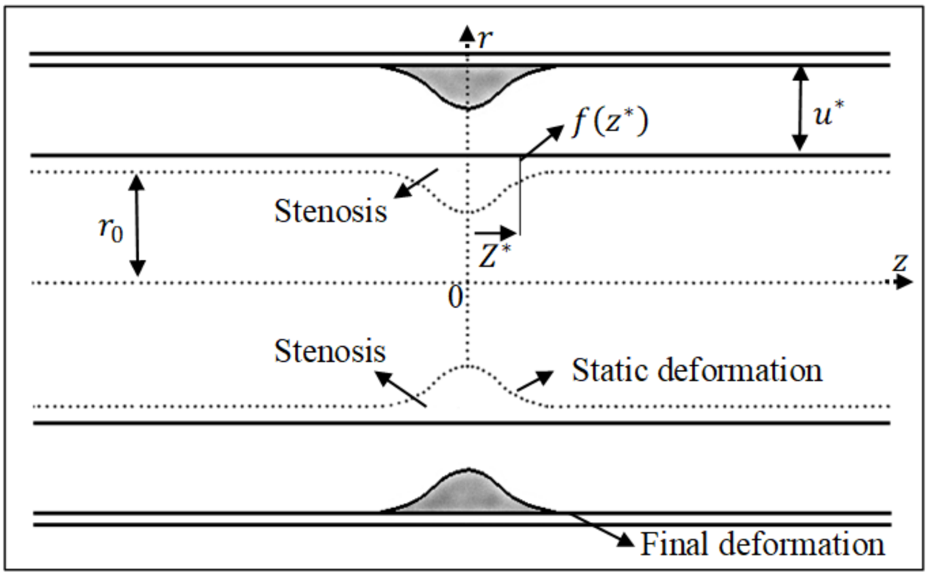


**Figure 1: Geometry of thin elastic tube with stenosis [5]**

where $\mu$ is the shear modulus of the material of tube, $\lambda_z$ is the axial stretch, $\lambda_1$ and $\lambda_2$ are the stretch ratios along the meridional and circumferential curves, $\Sigma$ is the strain energy density of the membrane, $R_0$ is the radius of circularly cylindrical tube, $z^*$ is the axial coordinate after static deformation, $f^*$ is a function that characterizes the axially symmetric stenosis on the surface of arterial wall, $u^*$ is the dynamical radial displacement, $P^*$ is the inner pressure applied by the fluid, $H$ is the thickness in the undeformed configuration, $r_0$ is the deformed radius at the origin of the coordinate system, $\rho_0$ is the mass density of tube, and $t^*$ is the time parameter.

## 2.2 Equation of Fluid

The blood is identified as the incompressible inviscid fluid. The equations of inviscid fluid [5] are expressed by

$$\frac{\partial w^*}{\partial t^*}+w^*\frac{\partial w^*}{\partial z^*}+\frac{1}{\rho_f}\frac{\partial P^*}{\partial z^*}=0 \qquad Eq.\ 2$$

$$2\frac{\partial u^*}{\partial t^*}+2w^*\left(-f^{*\prime}+\frac{\partial u^*}{\partial z^*}\right)+(r_0-f^*+u^*)\frac{\partial w^*}{\partial z^*}=0 \qquad Eq.\ 3$$

where $w^*$ is the averaged axial fluid velocity, $\rho_f$ is the mass density of fluid, and $P^*$ is the averaged fluid pressure.

## 2.3 Nondimensionalised Equations

Equations 1-3 are dimensional equations. The introduction of nondimensional quantities is convenient to convert the dimensional equations into nondimensional equations [5]. The nondimensionalised quantities introduced by Demiray [5] to eliminate the dimensional quantities are written as follows.

$$t^* = \left(\frac{R_0}{c_0}\right)t, \qquad z^* = R_0 z, \qquad u^* = R_0 u, \qquad m = \frac{\rho_0 H}{\rho_f R_0}, \qquad w^* = c_0 w,$$
$$f^* = R_0 f, \qquad r_0 = R_0 \lambda_\theta, \qquad P^* = \rho_f c_0^2 p, \qquad c_0^2 = \frac{\mu H}{\rho_f R_0} \qquad \textit{Eq. } 4$$

Equation 4 is utilized into Eq.1 – Eq. 3 by applying the chain rule. The nondimensionalized equations of tube and fluid are obtained as below,

$$p = \frac{m}{\lambda_z(\lambda_\theta - f + u)}\frac{\partial^2 u}{\partial t^2} + \frac{1}{\lambda_z(\lambda_\theta - f + u)}\frac{\partial \Sigma}{\partial \lambda_2} - \frac{1}{(\lambda_\theta - f + u)}\frac{\partial}{\partial z}\left\{\frac{\left(-f' + \frac{\partial u}{\partial z}\right)}{\left[1 + \left(-f' + \frac{\partial u}{\partial z}\right)^2\right]^{\frac{1}{2}}}\frac{\partial \Sigma}{\partial \lambda_1}\right\} \qquad \textit{Eq. } 5$$

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} = 0. \qquad \textit{Eq. } 6$$

$$2\frac{\partial u}{\partial t} + 2w\left(-f' + 2\frac{\partial u}{\partial z}\right) + (\lambda_\theta - f + u)\frac{\partial w}{\partial z} = 0. \qquad \textit{Eq. } 7$$

The stretched coordinates introduced by Demiray [5] is expressed as

$$\xi = \varepsilon^{\frac{1}{2}}(z - gt), \qquad \textit{Eq. } 8$$
$$\tau = \varepsilon^{\frac{3}{2}}z \qquad \textit{Eq. } 9$$

where $\varepsilon$ is a small parameter that measures the weakness of nonlinearity and dispersion, and $g$ is the scale parameter that will be obtained from the solution. From Eq.9, $z$ is solved in terms of $\tau$, which is

$$z = \varepsilon^{-\frac{3}{2}}\tau \qquad \textit{Eq. } 10$$

Introducing Eq. 9 into the expression for the function $f(z)$ to be

$$f(z) = h(\varepsilon, \tau). \qquad \textit{Eq. } 11$$

The differential relations are introduced as follows,

$$\frac{\partial}{\partial t} \to -\varepsilon^{\frac{1}{2}}g\frac{\partial}{\partial \xi}, \qquad \frac{\partial}{\partial z} \to \varepsilon^{\frac{1}{2}}\frac{\partial}{\partial \xi} + \varepsilon^{\frac{3}{2}}\frac{\partial}{\partial \tau}. \qquad \textit{Eq. } 12$$

By assuming that the function $h(\varepsilon, \tau)$ and the field variables $u$, $w$, and $p$ can be written as asymptotic series [5] as follows,

$$\begin{aligned} h &= \varepsilon h_1(\tau) + \varepsilon^2 h_2(\tau) + \cdots, \\ u &= \varepsilon u_1(\xi,\tau) + \varepsilon^2 u_2(\xi,\tau) + \cdots, \\ w &= \varepsilon w_1(\xi,\tau) + \varepsilon^2 w_2(\xi,\tau) + \cdots, \\ p &= p_0 + \varepsilon p_1(\xi,\tau) + \varepsilon^2 p_2(\xi,\tau) + \cdots. \end{aligned} \qquad \textit{Eq.}13$$

Now, introducing differential relations and expansion Eq. 13 into Eq. 5 – Eq. 7, various orders of differential equations are obtained.

$O(\varepsilon)$ equations:

$$-g\frac{\partial w_1}{\partial \xi} + \frac{\partial p_1}{\partial \xi} = 0, \qquad \textit{Eq. } 14$$

$$-2g\frac{\partial u_1}{\partial \xi} + \lambda_\theta\frac{\partial w_1}{\partial \xi} = 0, \qquad \textit{Eq. } 15$$

$$p_1 = \beta_1(u_1 - h_1). \qquad \textit{Eq. } 16$$

$\mathrm{O}(\varepsilon^2)$ equations:

$$-g\frac{\partial w_2}{\partial \xi}+w_1\frac{\partial w_1}{\partial \xi}+\frac{\partial p_2}{\partial \xi}+\frac{\partial p_1}{\partial \tau}=0, \qquad Eq.\ 17$$

$$-2g\frac{\partial u_2}{\partial \xi}+2w_1\frac{\partial u_1}{\partial \xi}+\lambda_\theta\frac{\partial w_2}{\partial \xi}+\lambda_\theta\frac{\partial w_1}{\partial \tau}-h_1\frac{\partial w_1}{\partial \xi}+u_1\frac{\partial w_1}{\partial \xi}=0, \qquad Eq.\ 18$$

$$p_2=\left(\frac{mg^2}{\lambda_\theta\lambda_z}-\alpha_0\right)\frac{\partial^2 u_1}{\partial \xi^2}+\beta_1(u_2-h_2)+\beta_2(u_1-h_1)^2. \qquad Eq.\ 19$$

## 2.4 Solution of the Field Equations

From the solution of Eq. 14 – Eq. 16, the following equations are obtained

$$u_1=U(\xi,\tau), \qquad Eq.\ 20$$

$$p_1=\frac{2g^2}{\lambda_\theta}(U-h_1). \qquad Eq.\ 21$$

$$w_1=\frac{2g}{\lambda_\theta}\ [U+\overline{w}_1(\tau)]. \qquad Eq.\ 22$$

Equations 20 – 22 are then introduced into Eq. 17 – Eq. 19 yields the following equations.

$$-g\frac{\partial w_2}{\partial \xi}+\frac{4g^2}{\lambda_\theta^2}\ [U+\overline{w}_1(\tau)]\frac{\partial U}{\partial \xi}+\frac{\partial p_2}{\partial \xi}+\frac{2g^2}{\lambda_\theta}\left(\frac{\partial U}{\partial \tau}-\frac{dh_1}{d\tau}\right)=0, \qquad Eq.\ 23$$

$$-2g\frac{\partial u_2}{\partial \xi}+\frac{4g}{\lambda_\theta}[U+\overline{w}_1(\tau)]\frac{\partial U}{\partial \xi}+\lambda_\theta\frac{\partial w_2}{\partial \xi}+2g\left(\frac{\partial U}{\partial \tau}+\frac{d\overline{w}_1}{d\tau}\right)-\frac{2g}{\lambda_\theta}h_1\frac{\partial U}{\partial \xi}+\frac{2g}{\lambda_\theta}U\frac{\partial U}{\partial \xi}=0, \qquad Eq.\ 24$$

$$p_2=\left(\frac{mg^2}{\lambda_\theta\lambda_z}-\alpha_0\right)\frac{\partial^2 U}{\partial \xi^2}+\beta_1(u_2-h_2)+\beta_2(U-h_1)^2. \qquad Eq.\ 25$$

Eliminate $w_2$ from Eq. 23 and Eq. 24 will obtain

$$-\frac{2g^2}{\lambda_\theta}\frac{\partial u_2}{\partial \xi}+\frac{\partial p_2}{\partial \xi}+\frac{4g^2}{\lambda_\theta}\frac{\partial U}{\partial \tau}+\frac{10g^2}{\lambda_\theta^2}U\frac{\partial U}{\partial \xi}+\frac{2g^2}{\lambda_\theta^2}(4\overline{w}_1-h_1)\frac{\partial U}{\partial \xi}+\frac{2g^2}{\lambda_\theta}\frac{d}{d\tau}(\overline{w}_1-h_1)\ =0. \qquad Eq.\ 26$$

Next, substitute Eq. 25 into Eq. 26 gives

$$\begin{aligned}\frac{4g^2}{\lambda_\theta}\frac{\partial U}{\partial \tau}+\left(\frac{10g^2}{\lambda_\theta^2}+2\beta_2\right)U\frac{\partial U}{\partial \xi}+\frac{\partial U}{\partial \xi}\left(\frac{8g^2}{\lambda_\theta^2}\overline{w}_1-\frac{2g^2}{\lambda_\theta^2}h_1+2\beta_2h_1\right)+\left(\frac{mg^2}{\lambda_\theta\lambda_z}-\alpha_0\right)\frac{\partial^3 U}{\partial \xi^3}\\+\frac{2g^2}{\lambda_\theta}\frac{d}{d\tau}(\overline{w}_1-h_1)=0.\end{aligned} \qquad Eq.\ 27$$

Thus, Eq. 27 must even be valid when $U=0$, which corresponds to steady flow. Thus, KdV equation with variable coefficients is written as

$$\frac{\partial U}{\partial \tau}+\mu_1U\frac{\partial U}{\partial \xi}+\mu_2\frac{\partial^3 U}{\partial \xi^3}+\mu_3h_1(\tau)\frac{\partial U}{\partial \xi}=0, \qquad Eq.\ 28$$

where the coefficients $\mu_1$, $\mu_2$ and $\mu_3$ are

$$\mu_1=\frac{5}{2\lambda_\theta}+\frac{\beta_2}{\beta_1}, \qquad \mu_2=\frac{m}{4\lambda_z}-\frac{\alpha_0}{2\beta_1}, \qquad \mu_3=\frac{3}{2\lambda_\theta}-\frac{\beta_2}{\beta_1}. \qquad Eq.\ 29$$

## 2.5 Progressive wave solution

An analytical solution for the KdV equation with variable coefficient is suggested in the following form [6]:

$$U = f(\zeta), \quad Eq.\ 30$$

$$\zeta = \beta[\xi - \varphi(\tau)], \quad Eq.\ 31$$

where $\beta$ is constant, and $\varphi(\tau)$ is unknown function of the variable $\tau$ which is determined from the solution of differential equation. Introducing Eq. 30 and Eq. 31 into Eq. 28 results in

$$\zeta = \left(\frac{\mu_1 c}{12\mu_2}\right)^{\frac{1}{2}}\left[\xi - \mu_3\int_0^\tau h_1(s)ds + \frac{1}{3}c\mu_1\tau\right]. \quad Eq.\ 32$$

According to Demiray [6], the progressive wave solution has a hyperbolic function in the form of

$$U = c\,\mathrm{sech}^2(\zeta), \quad Eq.\ 33$$

where $c$ represents the amplitude of wave, and the equation of wave speed is obtained as

$$v_p = \frac{d\tau}{d\xi} = \frac{1}{\mu_3 h_1(\tau) + \frac{1}{3}\mu_1 c}. \quad Eq.\ 34$$

2.6 Numerical Results

In this section, the subsequent analysis will need the explicit expressions of the coefficients of $\alpha_0$, $\beta_1$, and $\beta_2$. Thus, the constitutive equation for soft biological tissues shall be utilized as proposed by Demiray [7]. The constitutive equation is expressed as

$$\Sigma = \frac{1}{2\alpha}\left\{\exp\left[\alpha\left(\lambda_\theta^2 + \lambda_z^2 + \frac{1}{\lambda_\theta^2\lambda_z^2} - 3\right)\right] - 1\right\}. \quad Eq.\ 35$$

where $\alpha$ is the material constant.

The explicit expressions of the coefficients $\alpha_0$, $\beta_1$ and $\beta_2$ are given as follows [5]:

$$\alpha_0 = \frac{1}{\lambda_\theta}\left(\lambda_z - \frac{1}{\lambda_\theta^2\lambda_z^3}\right)F,$$
$$\beta_1 = \left[\frac{4}{\lambda_\theta^5\lambda_z^3} + 2\frac{\alpha}{\lambda_\theta\lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)^2\right]F,$$
$$\beta_2 = \left[-\frac{10}{\lambda_\theta^6\lambda_z^3} + \frac{\alpha}{\lambda_\theta\lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)\left(1 + \frac{11}{\lambda_\theta^4\lambda_z^2}\right) + 2\frac{\alpha^2}{\lambda_\theta\lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)^3\right]F. \quad Eq.\ 36$$

where $F$ is a function represented by

$$F = \exp\left[\alpha\left(\lambda_\theta^2 + \lambda_z^2 + \frac{1}{\lambda_\theta^2\lambda_z^2} - 3\right)\right]. \quad Eq.\ 37$$

## 3. Results and Discussion

In this section, a quantitative analysis towards the radial displacement, wave speed and the fluid pressure are presented. The value of the material constant, $\alpha$ is set as 1.948, which is extracted from Demiray [5], which compares the present model with the experimental measurements on canine abdominal artery by [8].

3.1 Radial Displacement

The wave amplitude, $c = 1$, the circumferential stretch, $\lambda_\theta = 1.45$, the axial stretch, $\lambda_z = 1.6$, the height of stenosis, $\delta = 0.01$, and $m = 0.1$ are fixed to obtain the graph of radial displacement. Figure 2 presents the variation of the radial displacement with the space, $\tau$, for different values of the time, $\xi$.

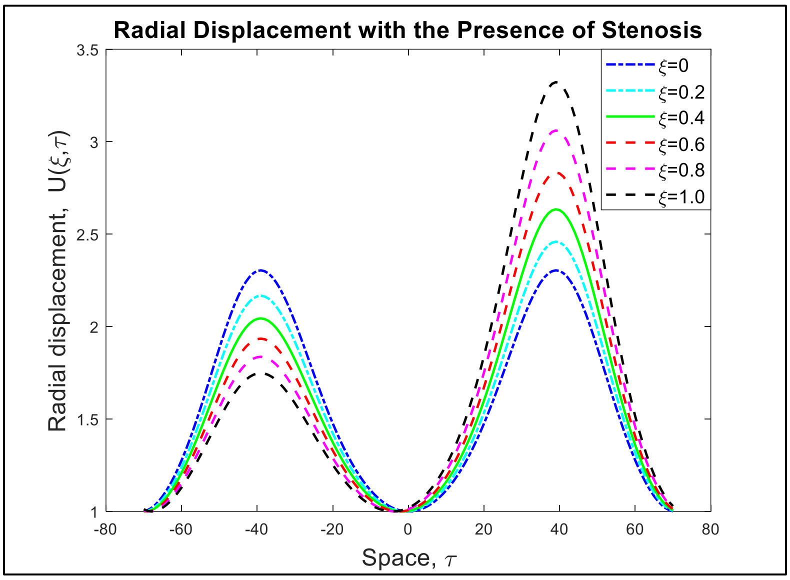


**Figure 2: Radial displacement, $U$, against space variable, $\tau$, for different time at $\delta = 0.01$ with the presence of stenosis.**

From Figure 2, the space at $\tau = 0$ indicates the location of the stenosis. The ranges $-80 < \tau < 0$ represents pre-stenosis and $0 < \tau < 80$ represents post-stenosis. The curves of wave on the left side represent the haemodynamic of blood before passes through the stenosis. The curves of wave on the right side represent the haemodynamic of blood after passes through the stenosis.

At the locations $\tau = -80$ until $\tau = 0$, it is noticed that the amplitude of wave decreases as the time increases. However, when the fluid passes through the stenosis, the opposite characteristics happens. After the fluid passes through the stenosis, when the time increases, the amplitude of wave increases.

When blood flow through stenosed artery, there exist kinetic energy, and this energy is determined by its amplitude and frequency. This means that the higher amplitude of wave has higher kinetic energy. In order for the lowest amplitude of wave to pass through the stenosis, it has to generate more energy. So, after the fluid passes through the stenosis, this wave becomes the highest amplitude of wave. As for the highest amplitude of wave, it requires minimum energy for the fluid to pass through the stenosis. Therefore, after it passes through the stenosis, it is the lowest amplitude of wave.

## 3.2 Wave Speed

The wave amplitude, $c = 1$, the circumferential stretch, $\lambda_\theta = 1.45$, the axial stretch, $\lambda_z = 1.6$, $m = 0.1$ are fixed and the different height of stenosis is set as 0.01, 0.02, and 0.03 to obtain the graph of wave speed. Figure 3 shows the wave speed over the space, $\tau$ at different height of stenosis, $\delta$.

Before the fluid passes through the stenosis, the lowest stenosis, $\delta = 0.01$ has the highest wave speed compared to other higher stenosis. However, when the fluid passes through the stenosis, the lowest stenosis, $\delta = 0.01$ reaches its maximum wave speed due to the minimum of resistance for blood flow in the stenosed artery. Therefore, it can be concluded that when the height of stenosis increases, the wave speed decreases.

The wave speed of the fluid (blood) is affected by the resistance in the artery. The greater the resistance, the slower the speed of wave. The highest stenosis that impacts the wave with greater resistance has the lowest wave speed as the wavelength is smallest compared to other height of stenosis and vice versa. Viscosity of an inviscid fluid is extremely low and can be considered negligible. There is no damping effect on the speed of wave. Therefore, before and after the fluid passes through the stenosis, the speed of wave is the same.

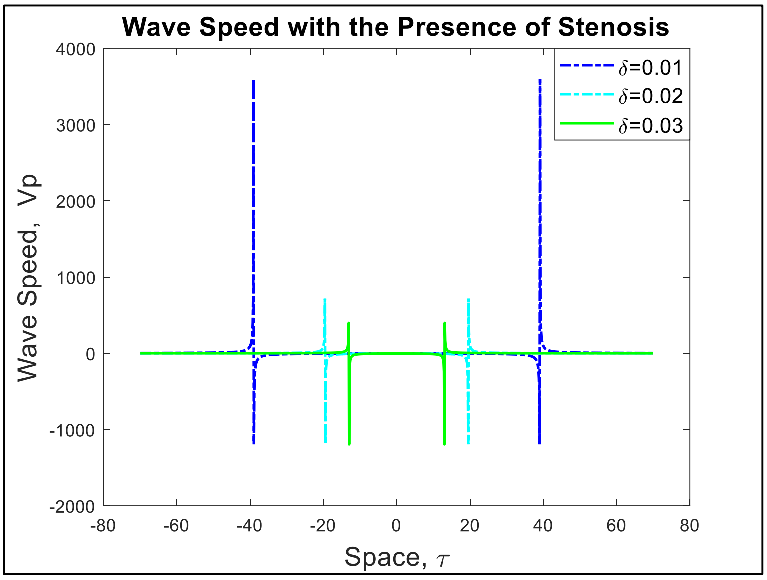


**Figure 3: Wave speed, $v_p$, against space variable, $\tau$, for different height of stenosis, $\delta$.**

3.3 Fluid Pressure

The wave amplitude, $c = 1$, the circumferential stretch, $\lambda_\theta = 1.45$, the axial stretch, $\lambda_z = 1.6$, the height of stenosis, $\delta = 0.01$, and $m = 0.1$ are fixed to obtain the graph of fluid pressure. Figure 4 illustrates the results for the fluid pressure over the space, $\tau$ at different time, $\xi$.

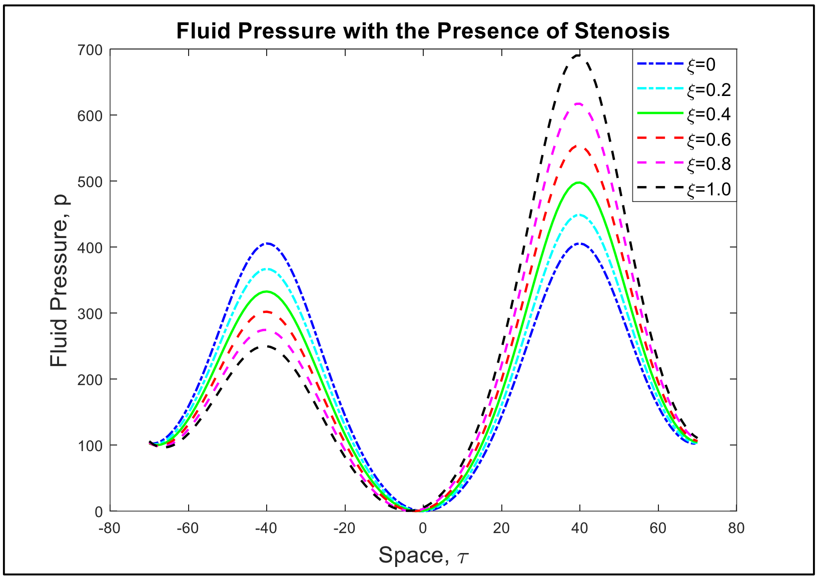


**Figure 4: Fluid pressure, $p$, against space variable, $\tau$, for different time at $\delta = 0.01$ with the presence of stenosis.**

Before the blood passes through the stenosis, the curves of wave are found to be decreasing. It means the fluid pressure decreases as time increases. However, the opposite characteristics happens when the fluid passes through the stenosis where when the time increases, the fluid pressure increases. This is due to the presence of energy in the fluid (blood). Since wave has energy, this shows that the higher the kinetic energy, the higher the fluid pressure.

In the wave, there exists kinetic energy. Before passing through the stenosis, the fluid pressure at time $\xi = 0.01$ has the lowest pressure, which is also the lowest amplitude of wave, needs greater energy to pass through the stenosis. Therefore, after the wave passes through the stenosis, the fluid becomes the highest pressure. As for the fluid pressure at time $\xi = 0$, has the highest amplitude of wave, it needs minimal energy for the fluid to pass through the stenosis, therefore is becomes the lowest pressure after the fluid passes through the stenosis.

3.4 Effect of Variation of Circumferential Stretch, $\lambda_\theta$, towards Wave Speed.

The wave amplitude, $c = 1$, the axial stretch, $\lambda_z = 1.6$, height of stenosis, $\delta = 0.5$, and $m = 0.1$ are fixed. The variation of circumferential stretch, $\lambda_\theta$ is set as 1.45, 1.55, and 1.65 to show the effect of variation of circumferential stretch, $\lambda_\theta$, towards wave speed. Figure 5 shows the graph of wave speed in different circumferential stretch, $\lambda_\theta$.

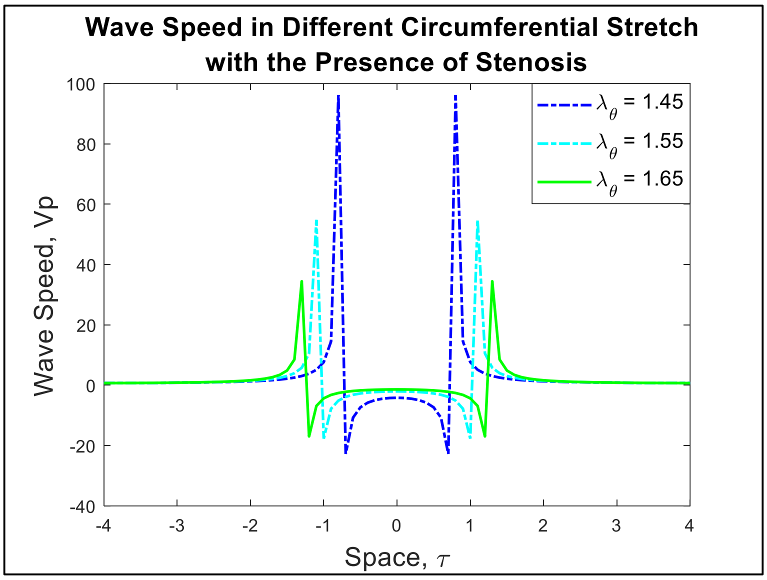


**Figure 5: Wave speed, $v_p$, against space variable, $\tau$, for different values of circumferential stretch, $\lambda_\theta$, at $\delta = 0.5$ with the presence of stenosis.**

It is shown that when the value of $\lambda_\theta$ increases, the speed of the wave decreases. The circumferential stretch with the highest value, $\lambda_\theta = 1.65$ will peak the earliest before passing through the stenosis, and after the fluid passes through the stenosis, the wave speed peaks last. As for the circumferential stretch with the lowest value, $\lambda_\theta = 1.45$ peaks last before the fluid passes through the stenosis, and peaks first after passing through the stenosis. Therefore, when the circumferential stretch increases, the wave speed decreases, and the peak-to-peak of the wave become wider.

The volume of blood flow is the same in different condition of circumferential. The dispersion effect causes the fluid (blood) to spread to a wider area and causes the wave speed to decrease. Thus, the smallest circumferential stretch, $\lambda_\theta = 1.45$, has the highest wave speed as there is lesser dispersion effect compared to the biggest circumferential stretch, $\lambda_\theta = 1.65$, which has the smallest wave speed.

## 4. Conclusion

In conclusion, the solitary wave propagation in a thin stenosed elastic tube filled with inviscid fluid has been studied using the reductive perturbation method. From the graphical output for radial displacement, the amplitude of the wave has inverse properties before and after the fluid passed through the stenosis, where the lowest amplitude of the wave before passing through the stenosis became the highest amplitude of the wave after passing through the stenosis and vice versa as time increased since this is due to the kinetic energy in the wave. For the wave speed, when the height of stenosis increased, the wave speed decreased, and the space between the peak-to-peak of the wave became smaller. The same properties were applied to the fluid pressure like the radial displacement. Lastly, the increase in circumferential stretch affected the wave speed to decrease, and the peak distance of the wave increased.

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