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Solving Higher Order Ordinary Differential Equation with Initial Condition using Differential Transform Method

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Abstract: In this study, higher order ordinary differential equation (ODE) with initial conditions have been solved using differential transform method (DTM). DTM is a semi-analytic numerical method for solving differential equations. Some examples are solved numerically, and the results are compared with adaptive polynomial method (APM), modification of Adomian decomposition method (MADM), Laplace transform method (LTM) and the exact solution. Maple 2021 have been used to solve the higher order equation. The obtained results are shown in tables and graphs to show that the DTM is more accurate compared to the others method.

Keywords: Differential Transformation Method, Higher Order Ordinary Differential Equation

1. Introduction

The differential transform method (DTM) is a semi-analytic numerical method based on Taylor series expansion for solving differential equations. DTM has been effectively used to a lot of problems, including linear and non-linear boundary value problems, as well as initial value problems. In the previous research, the bulk of researchers solved problems using numerical methods rather than analytical approaches, however most of the problems became difficult to solve. Furthermore, DTM has been commonly utilized as an alternate way in recent years since it is a semi-analytical method that solves problems using both numerical and analytical methods. Many mathematicians consider this method to be a fast convergence method that produces a succession of solutions that quickly converge to the approximate solution. For example, Hassan applied DTM to solve higher order initial value problems [1]. Next, solution of system of differential equations in a series solution using DTM have been studied by Ayaz [2]. In 1986, the concept of the DTM was the first to introduce to solve the problem of linear and nonlinear initial values in the analysis of electric circuit [3].

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The DTM, on the other hand, differs from standard higher-order Taylor series where the derivatives functions require additional computations. Therefore, this transformation method is an alternative way to obtain analytical solutions of differential equations. In this study we explored DTM to prove the effectively or ineffectively solving the higher order equation using several different examples. The purpose of this research is to use DTM to approximate the solution of higher order ODEs, which required less computational labour than other numerical approaches. The DTM also offers a higher level of accuracy and precision. It can be used to solve higher order (ODEs) and approximates exact solutions.

An ordinary differential equation (ODE) is a mathematical expression that contains with only one independent variable and one or more of its derivatives. Differential equation has been major part of pure and applied mathematics.

The following is the general form of higher order ODE is given as

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = f(x) \qquad Eq.1$$

with initial condition

$$y(x_0) = a, \quad y'(x_0) = b, \quad y''(x_0) = c, \quad \dots, \quad y^{(n)}(x_0) = d \qquad Eq.2$$

where a, b, c, d are may be integers, rational numbers, real numbers, complex numbers or more generally, members of any field.

2. Methodology

The differential transform of function $y(x)$ for the k^{th} derivative is defined as follows:

$$Y(k) = \frac{1}{k!}\left[\left|\frac{d^k\, y(x)}{dx^k}\right|_{x\,=\,0}\right] \qquad Eq.3$$

where $y(x)$ is the original function and $Y(k)$ is the transform function. The differential transform inverse of $Y(k)$ is defined as

$$y(x) = \sum_{k=0}^{\infty} Y(k)(x - x_0)^k \qquad Eq.4$$

The fundamental theorems on the differential transform method are as follows in Table 1.

Table 1: The fundamental theorems performed by differential transform method

Original functions	Transformed functions
$y(x) = u(x) \pm v(x)$	$Y(k) = U(k) \pm V(k)$
$y(x) = \alpha u(x)$	$Y(k) = \alpha U(k)$
$y(x) = y'(x)$	$Y(k) = (k+1)Y(k+1)$
$y(x) = y''(x)$	$Y(k) = (k+1)(k+2)Y(k+2)$
$y(x) = y^n(x)$	$Y(k) = (k+1)(k+2)\cdots(k+n)Y(k+n)$
$y(x) = u(x)v(x)$	$Y(k) = \sum_{1=0}^{k} V(1)U(r-1)$

$y(x) = x^m$	$Y(k) = \delta(k-m) = \begin{cases} 1 & \text{if } k = \text{m} \\ 0 & \text{if } k \neq \text{m} \end{cases}$
$y(x) = e^x$	$Y(k) = \dfrac{1}{k!}$
$y(x) = e^{\lambda x}$	$Y(k) = \dfrac{\lambda^k}{k!}$
$y(x) = (1+x)^m$	$Y(k) = \dfrac{m(m-1)\cdots(m-k+1)}{k!}$
$y(x) = \sin(jx+\alpha)$	$Y(k) = \dfrac{j^k}{k!}\sin\left(\dfrac{\pi k}{s}+\alpha\right)$
$y(x) = \cos(jx+\alpha)$	$Y(k) = \dfrac{j^k}{k!}\cos\left(\dfrac{\pi k}{s}+\alpha\right)$

3. Results and Discussion

Several examples of higher order equation are tested using DTM. The result obtained demonstrate the effectiveness of DTM for solving the higher order equations. Then, the results are compared with exact solution and others method.

Example 1: Consider the following third order equation as follows [5]:

$$S''' = e^x \quad Eq. 5$$

with initial conditions,

$$S(0) = 3, \qquad S'(0) = 1, \qquad S''(0) = 5 \quad Eq. 6$$

the exact solution is given,

$$S(x) = 2 + 2x^2 + e^x \quad Eq. 7$$

from initial condition and theorem, we obtain

$$Y(0) = 3, \qquad Y(1) = 1, \qquad Y(2) = \frac{5}{2} \quad Eq. 8$$

by applying Table 1, $Eq. 5$ is transformed into,

$$Y(k+3) = \frac{1}{(k+3)!} \quad Eq. 9$$

at $k = 0$,

$$Y(0+3) = \frac{1}{(0+3)!}$$

$$Y(3) = \frac{1}{6} \quad Eq. 10$$

substitute $k = 1,2,3,4$ into $Eq. 9$

$$Y(4) = \frac{1}{24}$$

$$Y(5) = \frac{1}{120}$$

$$Y(6) = \frac{1}{720}$$

$$Y(7) = \frac{1}{5040} \qquad Eq.11$$

by applying Taylor series,

$$Y(x) = 3 + x + \frac{5}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 \qquad Eq.12$$

Figure 1 shows the graph of numerical solution for exact solution, DTM and adaptive polynomial method (APM). Table 2 shows that numerical solution of DTM is near to exact solution. The table also showed that the absolute error increases as the value of x increases. Based on Table 3, the smallest of absolute error indicates the closeness between the method and the exact solution. It shows that the DTM is much accurate compared to APM.

Example 2: Consider the following fourth order equation as follows [6]:

$$y'''' + 2y''' = 4x - 8 \qquad Eq.13$$

with initial conditions,

$$y(0) = 0, \qquad y'(0) = 1, \qquad y''(0) = 0, \qquad y'''(0) = 1 \qquad Eq.14$$

the exact solution is given,

$$y(x) = x - \frac{1}{4}x^2 + \frac{5}{12}x^3 + \frac{1}{12}x^4 \qquad Eq.15$$

from initial condition and theorem, we obtain

$$Y(0) = 0, \qquad Y(1) = 1, \qquad Y(2) = 0, \qquad Y(3) = \frac{1}{6} \qquad Eq.16$$

by applying Table 1, $Eq.13$ is transformed into,

$$Y(k+4) = \frac{k!}{(k+4)!}\left[-2\left(\frac{(k+3)!}{k!} \times Y(k+3)\right) + 4\sum_{i=k}^{k} \delta(k-1) - 8\delta(k)\right] \qquad Eq.17$$

at $k = 0$,

$$Y(0+4) = \frac{0!}{(0+4)!}\left[-2\left(\frac{(0+3)!}{0!} \times Y(0+3)\right) + 4\sum_{i=0}^{0} \delta(0-1) - 8\delta(0)\right]$$

$$Y(4) = -\frac{1}{12} \qquad Eq.18$$

substitute $k = 1,2,3$ into $Eq.17$

$$Y(5) = \frac{1}{30}$$

$$Y(6) = -\frac{1}{90}$$

$$Eq.19$$

$$Y(7) = \frac{1}{315}$$

by applying Taylor series,

$$Y(x) = 0 + x + 0x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{30}x^5 - \frac{1}{90}x^6 + \frac{1}{315}x^7 \qquad Eq. 20$$

Figure 2 shows the graph of numerical solution for exact solution, DTM and modification of Adomian decomposition method (MADM). Table 4 shows the result of DTM accurate with exact solution. Based on Table 5, the smallest of absolute error indicates the closeness between the method and the exact solution. From graph, it shows that DTM is much accurate at $x \geq -0.1$ but at $x \leq -0.2$ it starts to diverge from exact solution. It can be concluded that for $x \leq -0.2$ MADM are more accurate.

Example 3: Consider the following fourth order equation as follows [7]:

$$y'''' = e^{-2x} + \sin(x) \qquad Eq. 21$$

with initial conditions,

$$y(0) = 0, \qquad y'(0) = 0, \qquad y''(0) = 0, \qquad y'''(0) = 0 \qquad Eq. 22$$

the exact solution is given,

$$y(x) = \frac{e^{-2x}}{16} - \frac{1}{16} - \frac{7}{8}x - \frac{1}{8}x^2 + \frac{1}{4}x^3 + \sin(x) \qquad Eq. 23$$

from initial condition and theorem, we obtain

$$Y(0) = 0, \qquad Y(1) = 0, \qquad Y(2) = 0, \qquad Y(3) = 0 \qquad Eq. 24$$

by applying Table 1, $Eq. 21$ is transformed into,

$$Y(k+4) = \frac{k!}{(k+4)!}\left[\left(\frac{1}{k!}\right)^{-2} + \frac{1^k}{k!}\sin\left(\frac{\pi k}{2!}\right)\right] \qquad Eq. 25$$

at $k = 0$,

$$Y(0+4) = \frac{0!}{(0+4)!}\left[\left(\frac{1}{0!}\right)^{-2} + \frac{1^0}{0!}\sin\left(\frac{\pi(0)}{2!}\right)\right]$$

$$Y(4) = \frac{1}{24} \qquad Eq. 26$$

substitute $k = 1,2,3$ into $Eq. 25$

$$Y(5) = \frac{1}{60}$$

$$Y(6) = \frac{1}{90}$$

$$Y(7) = \frac{43}{1008} \qquad Eq. 27$$

by applying Taylor series,

$$Eq. 28$$

$$Y(x) = \frac{1}{24}x^4 + \frac{1}{60}x^5 + \frac{1}{90}x^6 + \frac{43}{1008}x^7$$

Figure 3 shows the graph of numerical solution for exact solution, DTM and Laplace transform method (LTM). Table 6 shows the result of DTM accurate with exact solution. Based on Table 7, the smallest of absolute error indicates the closeness between the method and the exact solution. It shows that the LTM is much accurate compared to DTM. It can be concluded that LTM are accurate for any value of x.

Table 2: Numerical solution of DTM for example 1

x	Exact solution	DTM	Absolute Error
0	3.000000000	3.000000000	0.000000000
0.1	3.125170918	3.125170918	0.000000000
0.2	3.301402758	3.301402759	0.000000001
0.3	3.529858808	3.529858805	0.000000003
0.4	3.811824698	3.811824681	0.000000017
0.5	4.148721271	4.148721168	0.000000103
0.6	4.542118800	4.542118354	0.000000446
0.7	4.993752707	4.993751158	0.000001549
0.8	5.505540928	5.505536366	0.000004562
0.9	6.079603111	6.079591262	0.000011849
1	6.718281828	6.718253968	0.000027860

Table 3: Absolute error of the method used and minimum absolute error to exact solution for example 1

x	DTM	APM	Minimum
0	0.000000000	0.000000000	0.000000000
0.1	0.000000000	0.022654162	0.000000000
0.2	0.000000001	0.081133238	0.000000001
0.3	0.000000003	0.160986939	0.000000003
0.4	0.000000017	0.247464714	0.000000017
0.5	0.000000103	0.325515699	0.000000103
0.6	0.000000446	0.379788689	0.000000446
0.7	0.000001549	0.394632008	0.000001549
0.8	0.000004562	0.354093278	0.000004562
0.9	0.000011849	0.241919009	0.000011849
1	0.000027860	0.041553968	0.000027860

Table 4: Numerical solution of DTM for example 2

x	Exact Solution	DTM	Absolute Error
0	0.000000000	0.000000000	0.000000000
0.1	0.097925000	0.100158656	0.002233656
0.2	0.193466667	0.201209996	0.007743330
0.3	0.289425000	0.303898594	0.014473594
0.4	0.388800000	0.408834357	0.020034357
0.5	0.494791667	0.516517857	0.021726191
0.6	0.610800000	0.627362469	0.016562469
0.7	0.740425000	0.741714898	0.001289898
0.8	0.887466667	0.859875718	0.027590949
0.9	1.055925000	0.982121503	0.073803497
1	1.250000000	1.108730159	0.141269841

Table 5: Absolute error of the method used and minimum absolute error to exact solution for example 2

x	DTM	MADM	Minimum
0	0.000000000	0.000000000	0.000000000
0.1	0.002233656	0.001641667	0.001641667
0.2	0.007743330	0.008266667	0.007743330
0.3	0.014473594	0.022725000	0.014473594
0.4	0.020034357	0.048266667	0.020034357
0.5	0.021726191	0.088541667	0.021726191
0.6	0.016562469	0.147600000	0.147600000
0.7	0.001289898	0.229891667	0.001289898
0.8	0.027590949	0.340266667	0.027590949
0.9	0.073803497	0.483975000	0.073803497
1	0.141269841	0.666666667	0.141269841

Table 6: Numerical solution of DTM for example 3

x	Exact Solution	DTM	Absolute Error
0	0.000000000	0.000000000	0.000000000
0.1	0.000004089	0.000004349	0.000000260
0.2	0.000064334	0.000073257	0.000008923
0.3	0.000320934	0.000395429	0.000074495
0.4	0.001001403	0.001352737	0.000351334
0.5	0.002418004	0.003631882	0.001213879
0.6	0.004967112	0.008408571	0.003441460
0.7	0.009129997	0.017625674	0.008495677
0.8	0.015474623	0.034386895	0.018912272
0.9	0.024658090	0.063487438	0.038829348
1	0.037429440	0.112103175	0.074673735

Table 7: Absolute error of the method used and minimum absolute error to exact solution for example 3

x	DTM	LTM	Minimum
0	0.000000000	0.000000000	0.000000000
0.1	0.000000260	0.000000000	0.000000000
0.2	0.000008923	0.000000000	0.000000000
0.3	0.000074495	0.000000000	0.000000000
0.4	0.000351334	0.000000000	0.000000000
0.5	0.001213879	0.000000000	0.000000000
0.6	0.003441460	0.000000000	0.000000000
0.7	0.008495677	0.000000000	0.000000000
0.8	0.018912272	0.000000000	0.000000000
0.9	0.038829348	0.000000000	0.000000000
1	0.074673735	0.000000000	0.000000000

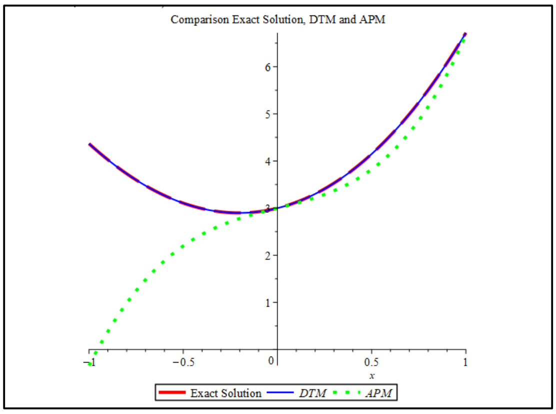

Figure 1: Comparison with Exact solution, DTM and APM

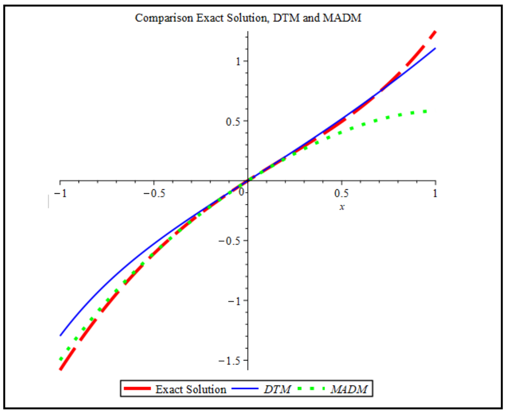

Figure 2: Comparison with Exact solution, DTM and MADM

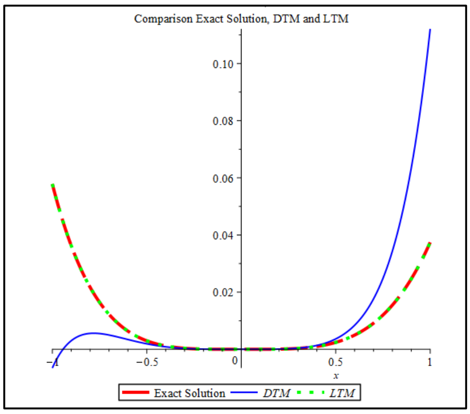

Figure 3: Comparison with Exact solution, DTM and LTM

4. Conclusion

From the result, it can conclude that DTM successfully solved higher order ODEs in example 1, example 2 and example 3. In this research, DTM is compared with APM, MADM, LTM and shows absolute error of DTM are much likely near to exact solution in example 1 and example 2 while example 3 absolute error of LTM is more accurate to exact solution. This demonstrates that DTM can solve higher order equation and it is the simplest method to apply. There are several recommendations for further study and research on DTM, including improving the algorithm of differential transform method to obtain exact solution and using modified differential transform method with Adomain polynomials or Laplace transform or Padé approximation to improve the solution of close to exact solution [8].

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