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Semi-Numerical Method for the Solution of System of Ordinary Differential Equation

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Abstract

In this study, systems of ordinary differential equations (ODEs) with initial conditions have been solved using differential transform method (DTM). Maple 2023 was employed to obtain approximate solutions, highlighting DTM's efficiency in reducing computational efforts and saving time. Comparative analyses with Adomian decomposition method (ADM) and fourth-order Runge-Kutta method (RK4) were conducted to evaluate DTM's accuracy. The results suggest that DTM's efficacy depends on equation characteristics and the number of terms used. A limited number of DTM terms are recommended for satisfactory approximations. DTM successfully solved one example, underscoring its simplicity and efficiency in handling differential problems with minimal computational workload.

1. Introduction

The differential transform method (DTM) is a useful tool developed by Zhou (1986) for estimating solutions to different types of equations. It has successfully solved problems involving both linear and nonlinear variables [1], providing a systematic and straightforward way to find approximations without using complicated numerical methods. DTM is versatile, applicable to various equations such as ordinary differential equations (ODEs), partial differential equations, and more. It stands out for its fast convergence rate and low computation error, making it effective for higher-order nonlinear equations and both initial and boundary value problems [2].

Hence, this method of transformation serves as an alternative approach for deriving analytical solutions to differential equations. In this investigation, we applied the DTM to assess its efficacy in solving ODE systems through various examples. The primary objective of this research is to employ DTM for approximating solutions to ODE systems, requiring comparatively less computational effort than other numerical methods. DTM stands out by providing heightened accuracy and precision, making it adept at solving ODE systems and approximating exact solutions.

Linear ODEs are generally simple to solve, and their solutions can be discovered using a variety of methods such as variable separation, Laplace transforms, and Fourier series. Previously, [3] solved the system by using RK4 and ADM. In addition, DTM offers an approximate solution with a fast convergence rate and a small computation error while solving differential equations problem. It can handle higher order non-linear differential equation, either initial value or boundary value problems [4].

The general form for linear system of ODEs is defined as follows:

$$\begin{aligned} x_1' &= f_1(t, x_1, x_2, ..., x_n) \\ x_2' &= f_2(t, x_1, x_2, ..., x_n) \\ &\vdots \\ x_n' &= f_n(t, x_1, x_2, ..., x_n) \end{aligned} \quad (1)$$

The $f_1, f_2, \cdots, f_n$ and the n functions are assumed to be known.

In conclusion, DTM proves to be a reliable method for solving systems of ordinary differential equations, offering a valuable alternative that reduces workload and potential errors during calculations. Other previous studies were done in mathematic application in worldwide according to various fields of studies [5, 6].

2. Methodology

The one-dimensional differential transform function is defined as follows [7]:

$$C(k) = \frac{1}{k!}\left[\frac{\partial^k}{\partial x^k} c(x)\right]_{x=0} \quad (2)$$

where $c(x)$ is the original function and $C(k)$ is the transformed function.

The differential inverse transform of $C(k)$ is defined as follows [8]:

$$c(x) = \sum_{k=0}^{\infty} C(k) x^k . \quad (3)$$

The fundamental operation performed by differential transformation as follow in Table 1.

Table 1 *The fundamental operation performed by differential transform [9].*

Original functions	Transformed functions
$c(t) = u(t) \pm v(t)$	$C(k) = U(k) \pm V(k)$
$c(t) = \alpha u(t)$	$C(k) = \alpha U(k)$
$c(t) = \frac{\partial}{\partial t} u(t)$	$C(k) = (k+1)U(k+1)$
$c(t) = \frac{\partial^r}{\partial t^r} u(t)$	$C(k) = (k+1)(k+2)...(k+r)U(k+r)$
$c(t) = u(t)v(t)$	$C(k) = \sum_{r=0}^{k} U(r)V(k-r)$
$c(t) = e^{\lambda t}$	$C(k) = \frac{\lambda^k}{k!}$
$c(t) = (1+t)^m$	$C(k) = \frac{m(m-1)...(m-k+1)}{k!}$
$c(t) = \sin(\omega t + \alpha)$	$C(k) = \frac{\omega^k}{k!} \sin\left(\frac{\pi k}{2!} + \alpha\right)$
$c(t) = \cos(\omega t + \alpha)$	$C(k) = \frac{\omega^k}{k!} \cos\left(\frac{\pi k}{2!} + \alpha\right)$

2.1 System of Linear Ordinary Differential Equations

Examples for system of linear ODEs are tested using DTM. The results obtained demonstrate the effectiveness of DTM for solving the system of linear ODEs. Then the result are compared with exact solution, fourth- order Rungge-Kutta (RK4), and Adomian Decomposition Method (ADM) [10].

Example 1: Consider the system of linear ordinary differential equations [3] as follow:

$$x' = x + y, \quad (4)$$

$$y' = -x + y, \quad (5)$$

with initial conditions,

$$\begin{aligned} x(0) &= 0, \\ y(0) &= 1. \end{aligned} \quad (6)$$

The exact solution is given,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^t \sin t \\ e^t \cos t \end{pmatrix}. \quad (7)$$

By applying DTM to (4) and (5) from Table 1, we will obtain as follows

$$\begin{aligned} X(k+1) &= \frac{X(k) + Y(k)}{k+1}, \\ Y(k+1) &= \frac{-X(k) + Y(k)}{k+1}. \end{aligned} \quad (8)$$

By applying Table 1, the initial conditions in (6) can be transformed as

$$\begin{aligned} X(0) &= 0, \\ Y(0) &= 1. \end{aligned} \quad (9)$$

Substituting $k = 0$ and the initial condition in (9) into (8) we obtain

$$X(0+1) = \frac{X(0) + Y(0)}{0+1}$$

$$X(1) = 1 \quad (10)$$

$$Y(0+1) = \frac{-X(0) + Y(0)}{0+1}$$

$$Y(1) = 1 \quad (11)$$

Continue substituting $k = 1$ to $k = 5$ and the initial conditions in (9) into (10) and (11) to obtain the third term and so on. To clarify the calculation for k is automatically limited to 5, due to the close proximity of final result with the previous research result [3].

Table 2 *Numerical solution by using DTM.*

k	$X(k+1)$	$Y(k+1)$
0	1	1
1	1	0
2	$\frac{1}{3}$	$-\frac{1}{3}$
3	0	$-\frac{1}{6}$

4	$-\frac{1}{30}$	$-\frac{1}{30}$
5	$-\frac{1}{90}$	0

Using solution from Table 2, combine all the terms to obtained and do the series solution of Taylor series,

$$x(t)=\sum_{k=0}^{\infty} X(k)t^k$$

$$=X(0)t^0+X(1)t^1+X(2)t^2+X(3)t^3+X(4)t^4+X(5)t^5+X(6)t^6+\dots \quad (12)$$

$$y(t)=\sum_{k=0}^{\infty} Y(k)t^k$$

$$=Y(0)t^0+Y(1)t^1+Y(2)t^2+Y(3)t^3+Y(4)t^4+Y(5)t^5+Y(6)t^6+\dots \quad (13)$$

Obtain the Taylor series such as

$$x(t)=t+t^2+\frac{1}{3}t^3-\frac{1}{30}t^5-\frac{1}{90}t^6 \quad (14)$$

$$y(t)=1+t-\frac{1}{3}t^3-\frac{1}{6}t^4-\frac{1}{30}t^5 \quad (15)$$

After that, compute numerical solution of DTM using Excel. The result obtained as in Table 3 and Table 4. Fig. 1 and Fig. 2 illustrate the graphical output of the results [11, 12].

3. Result and Discussion

The results obtained demonstrate the effectiveness of DTM for solving the system of ODE. The numerical solution obtained as follows:

Table 3 *The numerical solution and absolute error for x*

t	Exact Solution	DTM	|Exact - DTM|	RK4 [3]	|Exact- RK4|	ADM [3]	|Exact - ADM|
0.05	0.0525417	0.0525417	0	0.0525417	0	0.0525417	0
0.50	0.790439	0.790451	1.2E-05	0.790439	0	0.790436	3E-06
1.00	2.287355	2.288889	1.53E-03	2.28736	0	2.28733	3E-05
1.50	4.470462	4.495312	2.49E-02	4.47046	0	4.4704	6E-05
2.00	6.718849	6.888889	1.7E-01	6.71885	0	6.71872	1.3E-04

Table 3 show that numerical solution for RK4 is more accurate to exact solution rather than solution for DTM and ADM.

Table 4 *The numerical solution and absolute error for y*

t	Exact Solution	DTM	|Exact - DTM|	RK4 [3]	|Exact - RK4|	ADM [3]	|Exact - ADM|
0.05	1.049957	1.04996	0	1.04996	0	1.04996	0
0.50	1.446889	1.446875	1.5E-05	1.44689	0	1.44688	1E-05
1.00	1.468694	1.466667	2.02E-03	1.46869	0	1.46868	1E-05
1.50	0.317022	0.278125	3.89E-02	0.317021	1E-06	0.317013	9E-06
2.00	-3.07493	-3.4	3.25E-01	-3.07494	1E-05	-3.07488	5E-05

Based on Table 4, the smallest of absolute error indicates the closeness between the method and the exact solution.

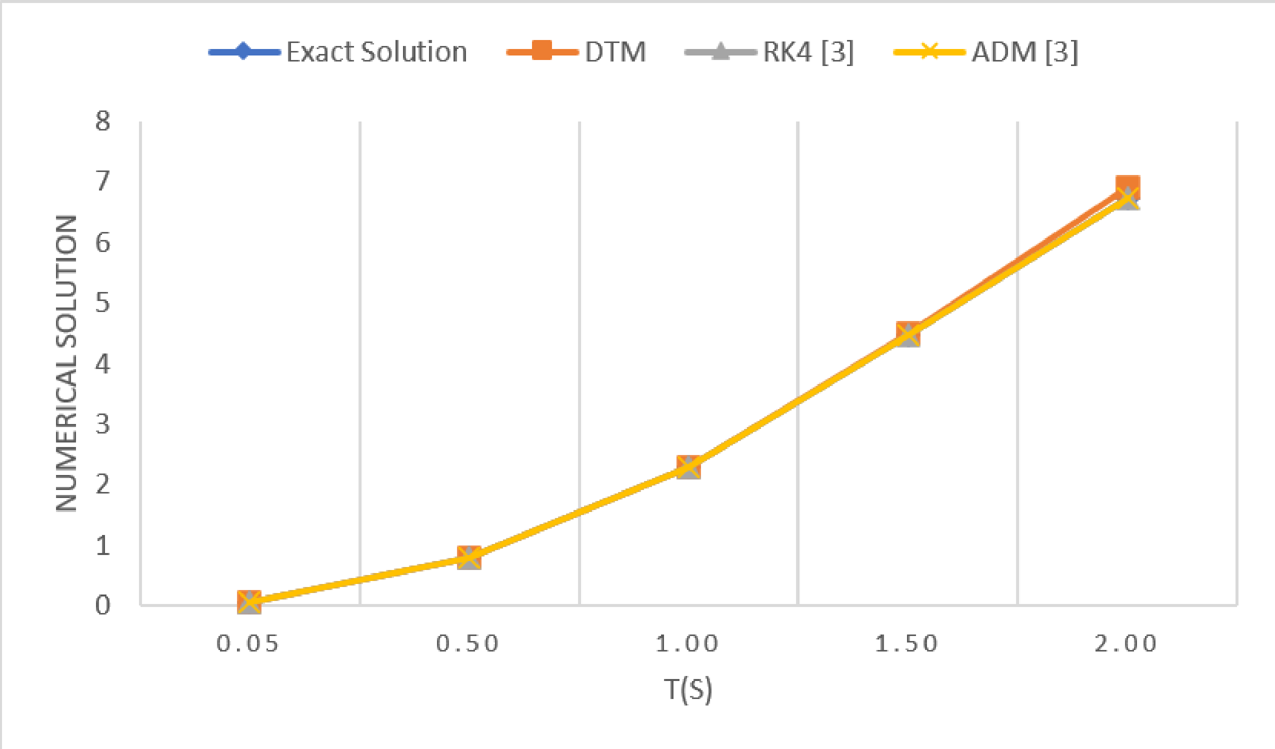

Fig. 1 *The exact, DTM, RK4, and ADM solution for x*

Based on Fig. 1 the DTM had compared with the exact solution, RK4, and ADM. The graph shows that DTM and RK4 have same solution with the exact solution. It can be concluded that DTM are accurate for any value of *x.*

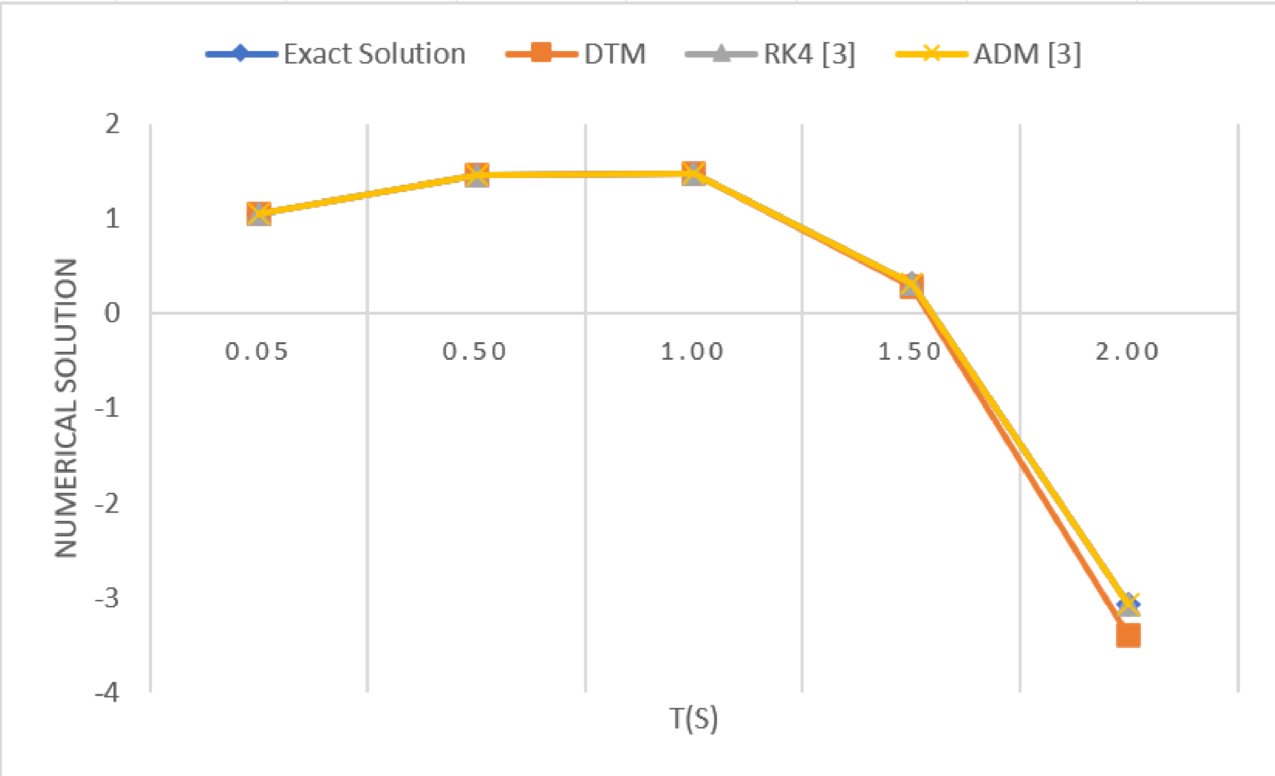

Fig. 2 *The exact, DTM, RK4, and ADM solution for y*

Fig. 2 shows the graph of numerical solution for exact solution, DTM, RK4, and ADM. From graph, it shows that DTM is much accurate at $t < 1.0$ but at $t > 1.5$ it starts to diverge from exact solution.

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Conclusion

From the result, it can be concluded that DTM successfully solved the system of ODEs. In this research, DTM was compared to RK4, ADM and show absolute error of DTM are much likely near to exact solution in *x* while *y* absolute error of RK4 is more accurate to exact solution. This research shows that DTM can solve the system or ODEs and it is the simplest method to apply. The are several recommendations for further study and research on DTM, including improving the algorithm of DTM to obtain exact solution and using modified DTM with Adomain polynomials or Laplace transform or Pade approximation to improve the solution of close to exact solution [9].

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of the paper.

Author Contribution

The authors confirm contribution to the paper as follows: ***study conception and design:*** *Zahimah Athirah Abdul Said:* ***Analyzed and interpreted data.*** *Zahimah Athirah Abdul Said and Noor Azliza Abd Latif. All authors reviewed the results and approved the final version of the manuscript.*

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