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A Logistic Growth Model with Optimization Matching Scheme for Chaotic Systems

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Abstract

A chaotic system is a deterministic nonlinear dynamic system but expresses unpredictable and random output. It is sensitive to initial conditions, where a slight change in input can produce significantly different outcomes. This paper describes a logistic growth model based on an optimization matching scheme to predict the solution of chaotic systems. Similarities and contradictions between chaotic systems and nonlinear systems are observed. Then, a loss function, which measures the differences between the chaotic system and the logistic growth model, is defined. Using a gradient method, the parameter in the logistic growth model is updated iteratively. Once convergence is achieved, the differences are minimized, and the optimal parameter is obtained. Hence, the solution of the logistic growth model with the optimal parameter approximates the solution of chaotic systems. For illustration, a one-dimensional two-parameter sin-cos (1DTPSC) system in encryption, a two-dimensional Van der Pol oscillator and a three-dimensional Chua's circuits are studied. These systems present chaotic behaviour for certain initial conditions and model parameters. The simulation results showed the accuracy of the logistic growth model in predicting the chaotic solutions. In conclusion, the applicability of the logistic growth model with an optimization matching scheme for handling chaotic systems is highly demonstrated.

1. Introduction

Mathematical modelling is a process that applies a mathematical model to describe real-world problems for making decisions, predictions or providing insights [1]. Dynamical systems have characteristics like nonlinear behaviour and changing over time [2]. Thus, dynamical systems are suited to formulate real-world problems in a mathematical model [3]. The nonlinearity of a dynamic system raises the issue of the existence of a solution, and even the chaotic behaviour of a nonlinear dynamic system renders its output unpredictable in the future. Hence, efficient computational methods have been developed to solve such nonlinear and chaotic systems.

Numerical methods, such as Euler and Runge-Kutta fourth-order methods, are applied to find the solution of nonlinear dynamical systems. The numerical solution is not the exact solution, but it can approximate the solution of a nonlinear dynamic system within an acceptable tolerance of numerical error [4]. A chaotic system is a nonlinear, deterministic dynamic system, but its output behaviour follows an irregular, unique solution. Since chaotic systems occupy randomness and disordered characteristics, thus the chaotic system is unstable, and the system output is hard to measure [5]. Moreover, the chaotic system is sensitive to an initial condition, where a slight change in the initial condition will bring the effect of a significant change in the system output [6].

Therefore, finding the solution to the chaotic systems motivates us to apply a logistic growth model based on an optimization matching scheme in studying chaotic systems. For this purpose, the differences between the chaotic system and the logistic growth model are measured. The unknown parameter in the logistic growth model is updated optimally using the gradient method to minimize the differences. Hence, the accuracy of the proposed method is guaranteed, and the optimal solution of the logistic growth model approximates the solution of the chaotic system at the end of convergence.

In addition, three objectives of the study are established. First, to identify the characteristics of chaotic systems so that the similarities and contradictions between chaotic systems and nonlinear systems are known. Second, to determine the solution of chaotic systems by solving the logistic growth model through the optimization matching scheme. Third, to improve the prediction solution by applying the previous actual solution to the analytical solution of the logistic growth model.

2. Materials and Methods

Consider a general continuous-time dynamical system [7] described by the differential equation,

$$\frac{dx}{dt} = f(x), \tag{1}$$

where $x \in \Re^n$ is the n-dimensional state variable of the system, dx/dt is the rate of change of the system over time t, and $f : \Re^n \times \Re \to \Re^n$ is the system function.

The discrete-time dynamical system to (1) is defined by

$$x_{k+1} = f(x_k), \tag{2}$$

where $x_k \in \Re^n$ is the current state of the system, and $x_{k+1} \in \Re^n$ is the next state of the system with k represents the time step.

Suppose the initial value of the state is $x(t_0) = x_0$, the solution of the systems (1) and (2) can be obtained numerically. However, these systems are nonlinear and sensitive to initial conditions, and their output is unpredictable and presents random behaviour, although there is no attending random noise in the system. These systems are known as chaotic systems. Hence, obtaining the solution of these systems is challenging.

2.1 Logistic Growth Model

Consider a logistic growth model in a differential equation [8], which is given as follows,

$$\frac{dx}{dt} = kx\left(1 - \frac{x}{K}\right), \tag{3}$$

where x is the population, k is known as the growth rate and K is the carrying capacity. While, dx/dt is the rate of changes over time t, and the initial population is $x(t_0) = x_0$ at the initial time t_0 with $t > t_0$.

Equation (3) can be written in a separable equation form,

$$\frac{1}{x\left(1 - \frac{x}{K}\right)} = k. \tag{4}$$

Then, do the partial fraction and integrating both sides of (4) to obtain

$$\int \left(\frac{1}{x} + \frac{1}{K - x}\right) dx = \int k dt. \tag{5}$$

Hence, the analytical solution to the model (3) is expressed by

$$x(t) = \frac{Kx_0}{x_0 + (K - x_0)e^{-k(t - t_0)}}. \tag{6}$$

2.2 Optimization matching scheme

Now, define an optimization problem [9] as follows,

$$J_{sse}(p) = r(p)^{\mathrm{T}} R r(p), \tag{7}$$

where

$$r(p) = f(x) - px(K - x) \tag{8}$$

is a model error, J_{sse} is the loss function, which represents the differences between the chaotic system and the logistic growth model, and R is the weight. Consider the first-order derivative of (7) with respect to p, that is,

$$\frac{\partial J_{sse}}{\partial p} = -x(K-x)R\big(f(x) - px(K-x)\big). \tag{9}$$

This derivative is the gradient to the loss function J_{sse}. Then, the parameter p can be updated by using the following recursion equation,

$$p^{(i+1)} = p^{(i)} - \alpha \times \left(\frac{\partial J_{sse}}{\partial p}\right)^{(i)} \tag{10}$$

with the given initial value $p(0) = p_0$ and step-size α. Equation (10) is known as gradient descent method.

Assume that p^* is the optimal parameter, which minimizes the loss function J_{sse} in (7). When the convergence is achieved, we have $p^* = p^{(i+1)} \approx p^{(i)}$, where the optimal parameter p^* is estimated satisfactorily. Thus, the analytical solution to the model (3) is given by

$$\hat{x}(t) = \frac{Kx_0}{x_0 + (K - x_0)e^{-p^*(t-t_0)}}. \tag{11}$$

Here, (11) shall give an approximation to the solution of the chaotic system (1). This approximate solution can be written by

$$\hat{x}(t_{i+1}) = \frac{Kx(t_i)}{x(t_i) + (K - x(t_i))e^{-p^*(t_{i+1}-t_i)}} \tag{12}$$

for $t_i < t < t_{i+1}$ during the calculation procedure.

Furthermore, to improve the accuracy of the prediction, we modify (12) as follows,

$$\hat{x}(t_{i+1}) = \frac{Ky(t_i)}{y(t_i) + (K - y(t_i))e^{-p^*(t_{i+1}-t_i)}} \tag{13}$$

for $t_i < t < t_{i+1}$, i = 0, 1, …, n, where y is the data-driven solution from a chaotic dynamic system. On this basis, the optimization matching (OM) scheme is formed.

3. Results and Discussion

This section examines three chaotic systems: one-dimensional image encryption, a two-dimensional Van der Pol oscillator, and a three-dimensional Chua circuit. An appropriate logistic growth model is proposed to handle these chaotic systems using the proposed OM scheme.

3.1 Chaotic image encryption

Consider a chaotic system, which is known as one-dimensional two-parameter sin-cos (1D-TPSC) encryption system [10], given as follows,

$$x_{n+1} = \beta \sin(\pi\mu(1 - x_n) + \cos(\pi x_n + 1)), \tag{14}$$

where μ and β are parameters with $\mu > 0$ and $\beta > 0$, whereas x_0 is the initial value and x has values ranging from $-\beta$ to β. The graphical result for (14) is shown in Figure 1 with $\mu = 8.8$ and $\beta = 3.6$ and $x_0 = 0.98461532023$.

Table 1 shows the simulation results for the 1D-TPSC system by applying the OM scheme. The initial parameter $p_0 = 0.4$ is employed in the logistic growth model before using the OM scheme, and the final parameter $p = 0.0324$ is the optimal parameter after convergence is achieved. Thus, the initial logistic growth model is given by

$$\frac{dx}{dt} = 0.4x(3.3021 - x) \tag{15}$$

with the exact solution

$$x(t_{i+1}) = \frac{3.3021x(t_i)}{x(t_i) + (3.3021 - x(t_i))\exp(-0.4(t_{i+1} - t_i))}. \tag{16}$$

Table 1 *Simulation result for single model parameter*

Initial Parameter	Final Parameter	Iteration Number	Elapsed Time (s)	Loss Function	Output MSE
0.4	0.032365	10	0.026813	2.1232	2.7498

While the final logistic growth model is

$$\frac{dx}{dt} = 0.0324x(3.3021 - x), \tag{17}$$

and the exact solution is

$$x(t_{i+1}) = \frac{3.3021x(t_i)}{x(t_i) + (3.3021 - x(t_i))\exp(-0.0324(t_{i+1} - t_i))}, \tag{18}$$

with the initial value $x(0) = 0.9846$. To improve the prediction solution, we modify the exact solution as follows,

$$x(t_{i+1}) = \frac{3.3021y(t_i)}{y(t_i) + (3.3021 - y(t_i))\exp(-0.0324(t_{i+1} - t_i))}. \tag{19}$$

Figure 2 shows the prediction solution using the final logistic growth model (17). It reveals that the solution curve does not fit closely to the solution of the system since the system presents chaotic characteristics. The prediction solution given by the improved solution (19), as shown in Figure 3, fits closely to the system solution and the chaotic characteristic is tracked accurately within an accepted tolerance. The prediction error using the improved solution is shown in Figure 4, where this prediction error is obviously smaller than the prediction error applying the final model. Hence, the prediction solution (19) is satisfactorily accepted with the output MSE of 4.7516×10^{-3}.

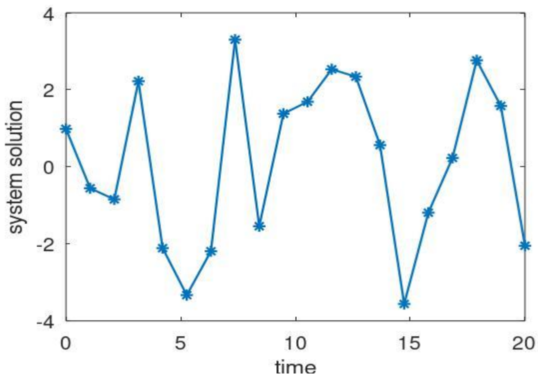


Fig. 1 *Solution of 1D-TPSC system*

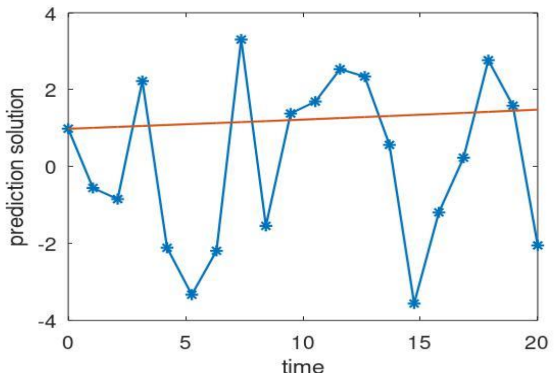


Fig. 2 *Prediction solution with final model and 1D-TPSC system solution*

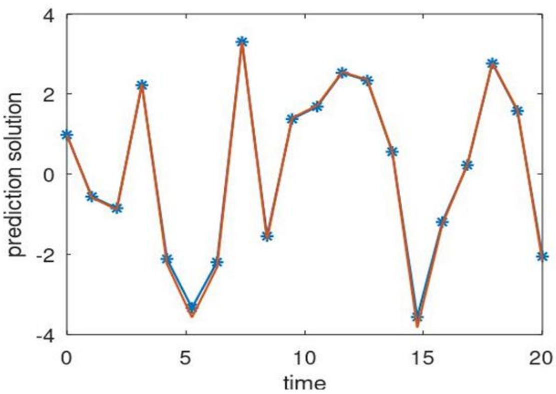


Fig. 3 *Prediction solution with improved model and 1D-TPSC system solution*

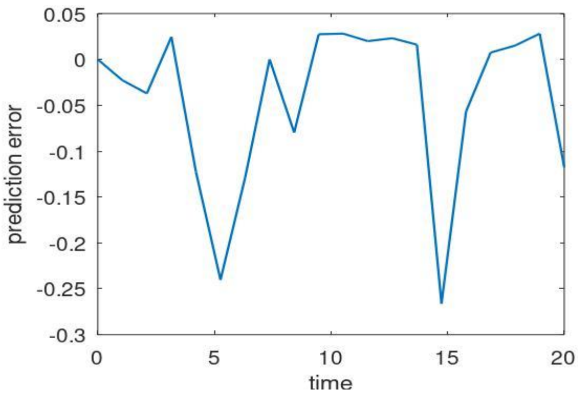


Fig. 4 *Prediction error using improved final model*

3.2 Van der Pol Oscillator

Consider an equation of Van der Pol oscillator [11], given below,

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = \beta\cos(\omega t), \tag{20}$$

where x is the solution of the oscillator, $\mu > 0$nd $\beta > 0$ are the model parameters and ω is the angular frequency of the external periodic force. This oscillator expresses the chaotic behaviour for $\mu = 5$, $\beta = 5$ and $\omega = 2.466$. The Van der Pol equation (20) can be written by

$$\frac{dx_1}{dt} = x_2 \tag{21}$$

$$\frac{dx_2}{dt} = -\mu(x_1^2 - 1)x_2 - x_1 + \beta\cos(\omega t) \tag{22}$$

The initial values are $x_1(0) = 1.6$ and $x_2(0) = 0.8$. Figure 5 shows the solution of the Van der Pol oscillator with chaotic motion.

Table 2 shows the simulation results from using two logistic differential equations after applying the OM scheme. The initial parameters $p_1 = 0.2$ and $p_2 = 0.1$ are used in the logistic differential equation before using the OM scheme, and the final parameter $p_1 = -3.3005$ and $p_2 = 0.4246$ are the optimal parameters after convergence is achieved. These initial logistic differential equations are given by

$$\frac{dx_1}{dt} = 0.2x_1(2.1322 - x_1) \tag{23}$$

$$\frac{dx_2}{dt} = 0.1x_2(7.4288 - x_2) \tag{24}$$

with the exact solutions,

$$x_1(t_{i+1}) = \frac{2.1322x_1(t_i)}{x_1(t_i) + (2.1322 - x_1(t_i))\exp(-0.2(t_{i+1} - t_i))}, \tag{25}$$

$$x_2(t_{i+1}) = \frac{7.4288x_2(t_i)}{x_2(t_i) + (7.4288 - x_2(t_i))\exp(-0.1(t_{i+1} - t_i))}, \tag{26}$$

and the initial conditions are $x_1(0) = 1.6$ and $x_2(0) = 0.8$. The final model of logistic differential equations has the exact solution,

$$x_1(t_{i+1}) = \frac{2.1322x_1(t_i)}{x_1(t_i) + (2.1322 - x_1(t_i))\exp(3.3005(t_{i+1} - t_i))}, \tag{27}$$

$$x_2(t_{i+1}) = \frac{7.4288x_2(t_i)}{x_2(t_i) + (7.4288 - x_2(t_i))\exp(-0.4246(t_{i+1} - t_i))}. \tag{28}$$

Table 2 *Simulation result for two model parameters*

Initial Parameter	Final Parameter	Iteration Number	Elapsed Time (s)	Loss Function	Output MSE
(0.2, 0.1)	(-3.3005, 0.4246)	52	0.759544	31.976	1.9240

Figure 6 shows the prediction solution of final models (25)-(26) and the Van der Pol system solution. The system solution exhibits chaotic behaviour, making it difficult to predict accurately. Thus, the predicted curves given by the final solutions (27)-(28) represent the best-fit solution for the system solution. To improve the accuracy of prediction, we modify the exact solution to be

$$x_1(t_{i+1}) = \frac{2.1322y_1(t_i)}{y_1(t_i) + (2.1322 - y_1(t_i))\exp(3.3005(t_{i+1} - t_i))}, \tag{29}$$

$$x_2(t_{i+1}) = \frac{7.4288y_2(t_i)}{y_2(t_i) + (7.4288 - y_2(t_i))\exp(-0.4246(t_{i+1} - t_i))}. \tag{30}$$

The prediction solutions using the improved solutions (29)-(30) are shown in Figure 7. The trend of the system solution is satisfactorily tracked, and the prediction errors are shown in Figure 8. The output MSE of using these improved solutions is 0.012038.

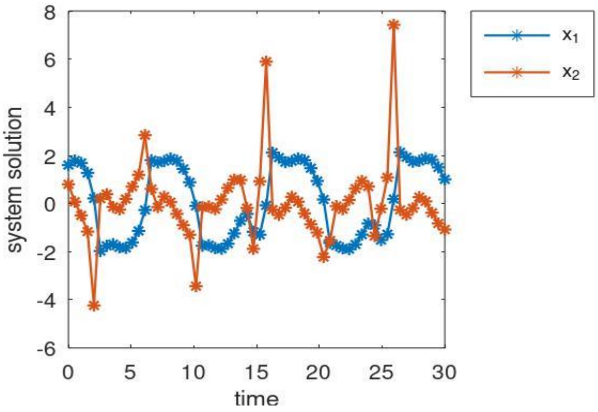


Fig. 5 *Solution of Van der Pol Oscillator*

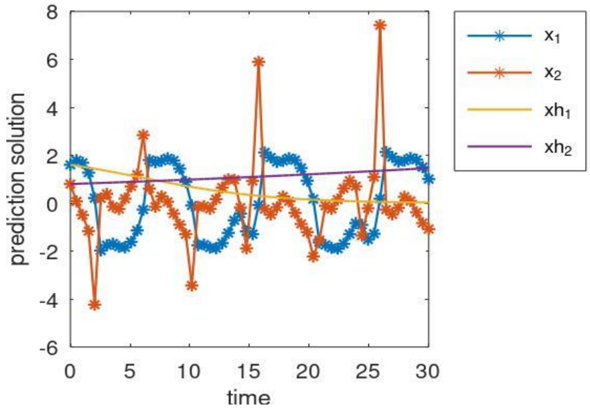


Fig. 6 *Prediction solution with two final models and Van der Pol system solution*

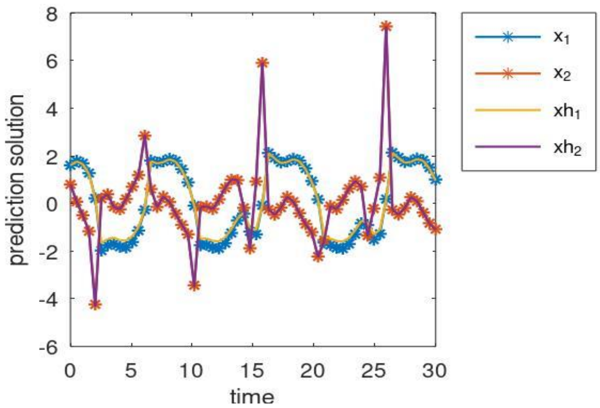


Fig. 7 *Prediction solution with two improved models and Van der Pol system solution*

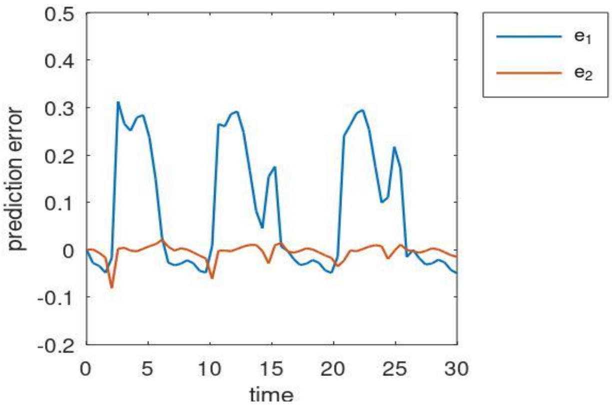


Fig. 8 *Prediction error using two improved models*

3.3 Chua's Circuit

Consider a Chua's circuit, which is a three-dimensional chaotic system [12], as follows

$$\frac{dx_1}{dt} = a(x_2 - x_1 - f(x_1)), \tag{31}$$

$$\frac{dx_2}{dt} = x_1 - x_2 + x_3, \tag{32}$$

$$\frac{dx_3}{dt} = -bx_2 \tag{33}$$

$$f(x_1) = m_1 x_1 + 0.5(m_0 - m_1)\left(|x_1 + 1| - |x_1 - 1|\right) \tag{34}$$

where a, b, m_0, m_1 are model parameters with the following values a = 15.6, b = 25.58, m_0 = -8/7, m_1 = -5/7. The initial conditions are $x_1(0)$ = 0.9, $x_2(0)$ = 0.01 and $x_3(0)$ = 0.5 for $0 < t < 30$. Figure 9 shows the dynamic behaviour of Chua's circuit.

Table 3 shows the simulation results for using three logistic differential equations to predict the solution of Chua's circuit in (31)-(33). The initial parameters p_1 = 0.8, p_2 = 0.5, p_3 = 0.2 are used in the logistic differential equations before running the OM scheme, and the final parameters p_1 = -345.872, p_2 = -19.002, p_3 = -140.873 are the optimal parameters after convergence is achieved. These initial models are given by

$$\frac{dx_1}{dt} = 0.8x_1(2.3243 - x_1)\,, \tag{35}$$

$$\frac{dx_2}{dt} = 0.5x_2(0.4323 - x_2)\,, \tag{36}$$

$$\frac{dx_3}{dt} = 0.2x_3(3.6915 - x_3)\,. \tag{37}$$

The exact solutions for these initial models are provided as follows,

$$x_1(t_{i+1}) = \frac{2.3243x_1(t_i)}{x_1(t_i) + (2.3243 - x_1(t_i))\exp(-0.8(t_{i+1} - t_i))}\,, \tag{38}$$

$$x_2(t_{i+1}) = \frac{0.4323x_2(t_i)}{x_2(t_i) + (0.4323 - x_2(t_i))\exp(-0.5(t_{i+1} - t_i))}\,, \tag{39}$$

$$x_3(t_{i+1}) = \frac{3.6915x_3(t_i)}{x_3(t_i) + (3.6915 - x_3(t_i))\exp(-0.2(t_{i+1} - t_i))}\,. \tag{40}$$

Table 3 *Simulation result for three model parameters*

Initial Parameter	Final Parameter	Iteration Number	Elapsed Time (s)	Loss Function	Output MSE
(0.8, 0.5, 0.21)	(-345.872, -19.002, -140.873)	961	11.7392	1878.0	2.6667

The final models have the following exact solutions,

$$x_1(t_{i+1}) = \frac{2.3243x_1(t_i)}{x_1(t_i) + (2.3243 - x_1(t_i))\exp(345.8(t_{i+1} - t_i))}\,, \tag{41}$$

$$x_2(t_{i+1}) = \frac{0.4323x_2(t_i)}{x_2(t_i) + (0.4323 - x_2(t_i))\exp(19.0(t_{i+1} - t_i))}\,, \tag{42}$$

$$x_3(t_{i+1}) = \frac{3.6915x_3(t_i)}{x_3(t_i) + (3.6915 - x_3(t_i))\exp(140.8(t_{i+1} - t_i))}\,. \tag{43}$$

Figure 10 shows the prediction solution given by the exact solution (41)-(43). It is noticed that the predicted curves represent the best-fit solution for the system solution.

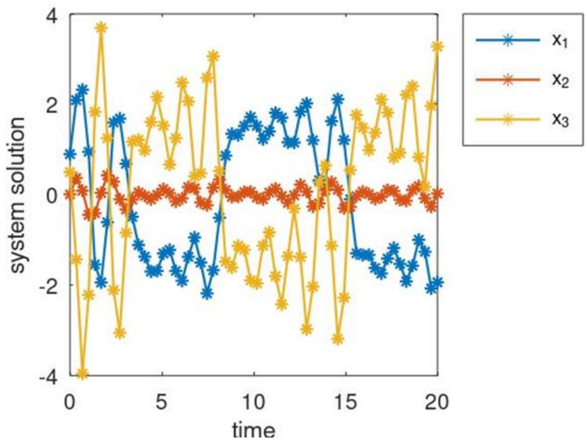


Fig. 9 *Solution of Chua's circuit system*

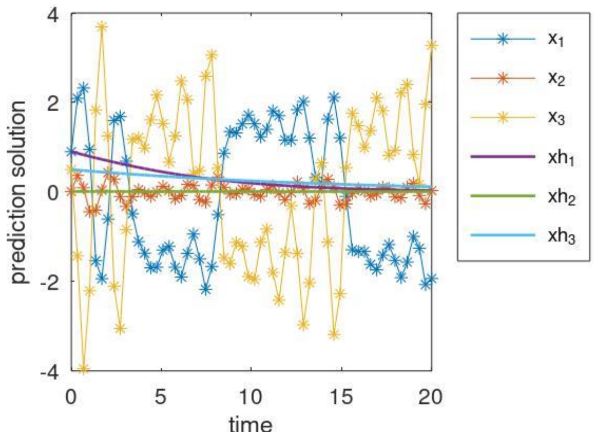


Fig. 10 *Prediction solution with three final models and Chua's circuit system solution*

To improve the accuracy of prediction, we modify the exact solution as below,

$$x_1(t_{i+1}) = \frac{2.3243y_1(t_i)}{y_1(t_i) + (2.3243 - y_1(t_i))\exp(345.8(t_{i+1} - t_i))}\,, \tag{44}$$

$$x_2(t_{i+1}) = \frac{0.4323y_2(t_i)}{y_2(t_i) + (0.4323 - y_2(t_i))\exp(19.0(t_{i+1} - t_i))}\,, \tag{45}$$

$$x_3(t_{i+1}) = \frac{3.6915y_3(t_i)}{y_3(t_i) + (3.6915 - y_3(t_i))\exp(140.8(t_{i+1} - t_i))}\,. \tag{46}$$

Figure 11 shows the prediction solution using the improved solutions (44)-(46). The prediction error is shown in Figure 12. Using these improved solutions gives the output MSE of 6.7690×10^{-3}. The chaotic attractors for Chua's circuit are shown in Figure 13.

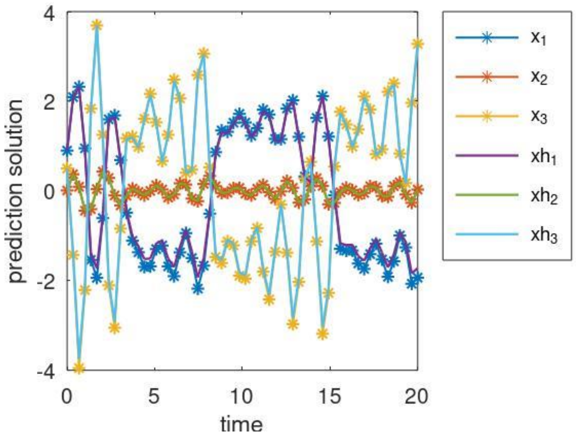


Fig. 11 *Prediction solution with three improved models and Chua's circuit system solution*

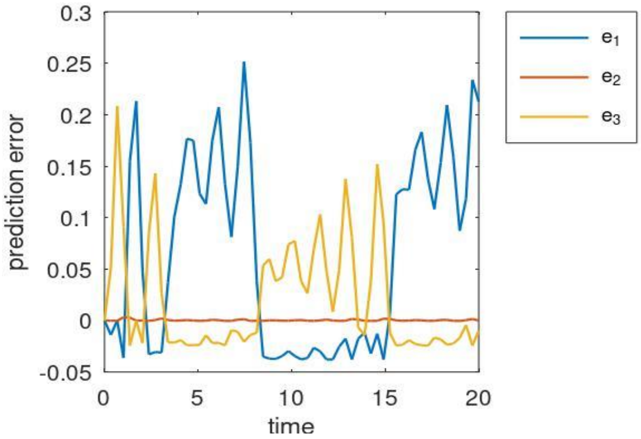
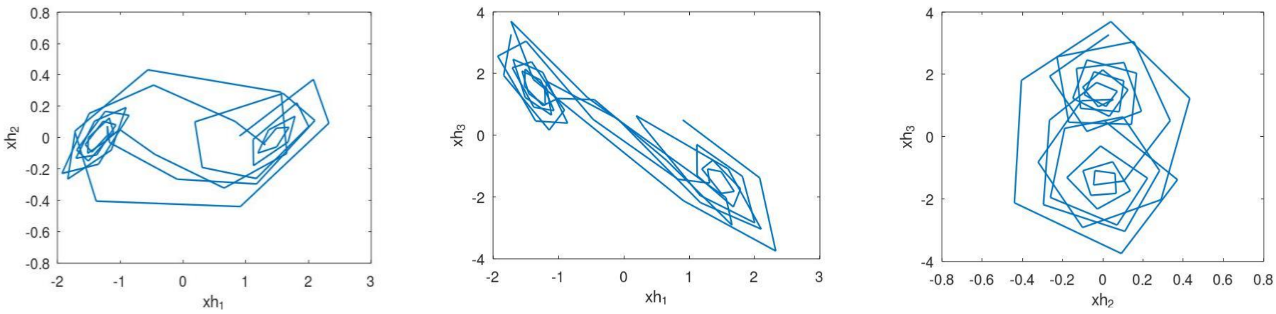


Fig. 12 *Prediction error using three improved models*



(a) $\hat{x}_1$ *versus* $\hat{x}_2$ (b) $\hat{x}_1$ *versus* $\hat{x}_3$ (c) $\hat{x}_2$ *versus* $\hat{x}_3$

Fig. 13 *Chaotic attractors for Chua's circuit.*

4. Conclusion

This paper discussed the logistic growth model with the optimization matching scheme to predict the solution of chaotic systems. During the iterative calculation, the loss function was minimized once the convergence was achieved, and the solution of the logistic growth model would approximate the solution of chaotic systems. For illustration, three chaotic systems, namely the 1D-TPSC encryption system, Van der Pol oscillator and Chua's circuit, were studied. The simulation results showed satisfactory solutions in predicting these chaotic systems, where the prediction accuracy was verified through the loss function and the output MSE. In conclusion, the optimization matching scheme demonstrates the efficiency in handling the solution of chaotic systems. For future research, it is recommended to explore the application of the proposed method for other more complicated chaotic systems.

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Conflict of Interest

The authors declare that there is no conflict of interests regarding the publication of the paper.

Author Contribution

The authors confirm contribution to the paper as follows: ***study conception and design:*** *Chong Hui Kuan, Kek Sie Long;* ***analysis and interpretation of results:*** *Chong Hui Kuan;* ***validation of results:*** *Kek Sie Long;* ***draft manuscript preparation:*** *Chong Hui Kuan. All authors reviewed the results and approved the final version of the manuscript. They have agreed to be accountable for all aspects of the work, ensuring that questions related to the accuracy or integrity of any part of the work are appropriately investigated and resolved.*

References

[1] Hartono, Y. (2020). Mathematical modelling in problem solving. *Journal of Physics: Conference Series, 1480(1),* 012001. https://doi.org/10.1088/1742-6596/1480/1/012001

[2] Irwin, M., & Wang, Z. (2017). Dynamic Systems Modeling. *The International Encyclopedia of Communication Research Methods,* 1–12. https://doi.org/10.1002/9781118901731.iecrm0074

[3] Rajendra, P., & Brahmajirao, V. (2020). Modeling of dynamical systems through deep learning. *Biophysical Reviews, 12(6),* 1311–1320. https://doi.org/10.1007/s12551-020-00776-4

[4] Courtney Remani. (2013). Numerical Methods for Solving Systems of Nonlinear Equations. https://www.lakeheadu.ca/sites/default/files/uploads/77/docs/RemaniFinal.pdf

[5] Boeing, G. (2016). Visual analysis of nonlinear dynamical systems: chaos, fractals, Self-Similarity and the Limits of Prediction. *Systems, 4(4),* 37. https://doi.org/10.3390/systems4040037

[6] Dokoumetzidis, A., Iliadis, A., & Macheras, P. (2001). Nonlinear dynamics and chaos theory: concepts and applications relevant to pharmacodynamics. *Pharmaceutical Research, 18(4),* 415–426. https://doi.org/10.1023/a:1011083723190

[7] O.Knil (2021). CONTINUOUS DYNAMICAL SYSTEMS I. https://abel.math.harvard.edu/archive/21b_fall_03/handouts/dynsys.pdf

[8] Renshaw, E. (1991). Modelling Biological Populations in Space and Time. *Cambridge University Press,* 6–9. https://doi.org/10.1017/CBO9780511624094

[9] Cheng, Z., Wang, W., Dai, Y., & Li, L. (2022). Novel One-Dimensional Chaotic System and Its Application in Image Encryption. *Complexity, 2022,* 1–16. https://doi.org/10.1155/2022/1720842

[10] Steeb, W., Huang, J. C., & Gou, Y. S. (1989). Comment on the Chaotic Behaviour of van der Pol Equations with an External Periodic Excitation. *Zeitschrift Für Naturforschung, 44(2),* 160–162. https://doi.org/10.1515/zna-1989-0211

[11] Agrawal, A. (2018, June 17). Loss functions and optimization algorithms. demystified. *Medium*. https://medium.com/data-science-group-iitr/loss-functions-and-optimization-algorithms-demystified-bb92daff331c

[12] Lv, E. (2021). CHUA's circuit simulation experiment based on saturation function. *Journal of Physics, 2078(1),* 012026. https://doi.org/10.1088/1742-6596/2078/1/012026