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Comparative Analysis of Numerical Methods for Earth Orbit Calculations in Celestial Mechanics

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Abstract

Numerical methods are indispensable for simulating celestial mechanics, specifically Earth's orbit around the Sun. These calculations are vital for understanding orbital dynamics, influencing satellite trajectories, seasonal patterns, and climate prediction. By leveraging numerical methods, researchers can analyse complex gravitational interactions and validate fundamental principles like Kepler's Laws. However, many numerical methods suffer from accuracy limitations, leading to errors in long-term orbital calculations, which can affect practical applications like satellite operations and space missions. This study compares Euler's method and the Fourth Order Runge-Kutta (RK4) method to validate Kepler's First Law, which posits that planets follow elliptical trajectories. MATLAB simulations were employed to implement these methods, and a convergence test was conducted to evaluate their accuracy. The findings reveal that while Euler's method offers simplicity, it accumulates significant numerical errors, leading to deviations in trajectory calculations. Conversely, RK4 method demonstrates superior precision and efficiency, accurately simulating Earth's elliptical orbit with minimal error. These results underscore the critical role of higher-order numerical methods in modelling orbital dynamics, offering valuable insights for applications in celestial mechanics and astrodynamics.

1. Introduction

Numerical simulations are essential for understanding how celestial bodies move, allowing us to model complex systems like planetary orbits. By studying Earth's orbit around the Sun, we can confirm key principles like Kepler's First Law while also gaining practical insights into predicting climate patterns, managing satellite operations, and exploring space. This study compares two numerical methods, Euler's method and the Fourth Order Runge-Kutta (RK4) method to simulate Earth's orbit. Through convergence tests, we assess their accuracy and efficiency to determine which method is better suited for celestial mechanics applications. This study seeks to validate Kepler's first law by visualising the Earth's elliptical orbit while implementing numerical methods such as Euler's method and the RK4 method to simulate the Earth's orbit. Furthermore, it aims to analyse and compare the accuracy and efficiency of these methods through a rigorous convergence test. By addressing these objectives, the study contributes to advancing the precision of orbital simulations and enhancing our understanding of numerical methods in celestial mechanics.

Kepler's laws of planetary motion are fundamental to our understanding of orbital dynamics. Introduced by Johannes Kepler in the early 17th century, these laws describe planetary movements with remarkable clarity. Kepler's first law, known as the Law of Ellipses, transformed the way we think about planetary motion by

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replacing the outdated circular orbit model with elliptical paths. His second law, the Law of Equal Areas, explains why planets speed up when closer to the Sun and slow down when farther away. The third law, the Law of Harmonies, establishes a precise mathematical relationship between a planet's orbital period and its average distance from the Sun [1]. Together, these laws offer a solid foundation for understanding the gravitational forces shaping our solar system.

To solve the equations that describe these motions, numerical methods like Euler's method and RK4 method are indispensable. Euler's method is straightforward and easy to use but tends to accumulate errors over long-term calculations [2], [3]. On the other hand, the RK4 method, a more advanced technique, minimizes these errors, making it a more reliable option for precise orbital simulations [2], [4].

Accurate models of celestial mechanics are crucial. Understanding Earth's orbit, for instance, helps predict seasonal changes [5], [6], improves satellite placement and management [7],[8], and supports the success of interplanetary missions [6], [9]. However, while much research has explored numerical methods for solving differential equations, detailed comparisons of Euler's method and RK4 method in simulating Earth's orbit are still limited. This study addresses that gap by evaluating these methods and highlighting their respective strengths and weaknesses [3], [10].

Outer space remains a vast and mysterious frontier, inspiring countless scientists to explore its depths. Earth, as our home, is a critical focus of this exploration, especially its orbit. By understanding Earth's orbit, we can forecast seasonal and climate patterns, which are heavily influenced by how solar energy is distributed. Moreover, accurate orbital knowledge is vital for satellite operations and space missions, ensuring that satellites remain stable, functional, and free from collisions. Despite the growing importance of numerical methods in addressing complex problems in science and engineering, there is a need for comprehensive studies comparing methods like Euler's method and RK4 method for tackling orbital dynamics. This research aims to fill that gap by analysing the performance and accuracy of these methods in simulating Earth's orbit.

By achieving these objectives, this study not only advances our theoretical understanding of celestial mechanics but also enhances practical applications in numerical modelling. The findings highlight the importance of precision in simulations and provide valuable insights for future research and real-world applications in astrodynamics and related fields.

2. Methodology

The methodology section provides a structured approach to evaluating the effectiveness of Euler's method and RK4 method in simulating Earth's orbit. It begins with a mathematical formulation of orbital dynamics, followed by a detailed explanation of the numerical methods. Finally, the setup for simulations and convergence testing is described.

2.1 Problem Formulation

Newton's law of gravity, a specific solution to the gravitational problem between two masses is described by Kepler's laws. According to this law, the gravitational force between two masses is proportional to their product and inversely proportional to the square of their distance. This force is given as below.

$$F = \frac{Gm_1m_2}{r^2} \tag{1}$$

where F is the force of attraction, m_1 and m_2 are the masses, r is the distance between two masses and G is the constant of universal gravitation.

Newton's law of motion says that the body's rate of change in momentum has a direction in which the force acts and is proportionate to the impressed force. We can use fundamental mathematical concepts to explain this. Therefore, let $\vec{r}$ represent mass's position vector. Next,

$$\vec{V} = \frac{d\vec{r}}{dt} \tag{2}$$

where $\vec{V}$ the velocity vector of the mass m**,**

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d^2\vec{r}}{dt^2} \tag{3}$$

The mass acceleration vector is denoted by $\vec{a}$. Let the external force acting on mass m be denoted by F. Next,

$$\vec{F} = m\frac{d^2\vec{r}}{dt^2} \tag{4}$$

This is how Newton's law is expressed in calculus.

The gravitational force between two masses, such as the Sun and Earth, is used to compute Earth's orbital speed around the Sun. Fig. 1 illustrates the gravitational force between the Sun, m_1 and Earth, m_2 where the force vectors act between the two masses, governed by Newton's Law of Gravitation. The force F_{m_1} and F_{m_2} are acting on Earth and Sun respectively.

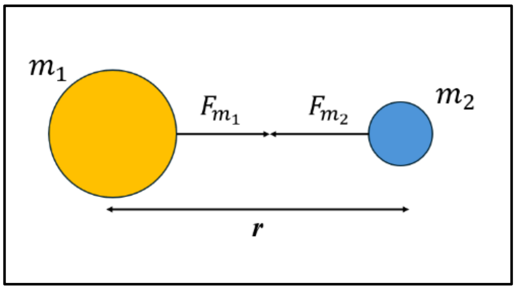

Fig. 1 Gravitational Force Between Two Masses

The gravitational force acting between the Sun and Earth is central to determining Earth's orbital path. The forces depicted in Fig. 1 align with the equations of motion derived from Newton's Laws, which serve as the foundation for simulating the orbit using numerical methods.

Let F stand for the force vectors acting on the mass m_2, and let $\vec{a}$ stand for the acceleration vector. Next, consider m_2 accelerated by gravity force. Thus,

$$m_2\vec{a} = -\vec{F} \tag{5}$$

From (1), (3), (4) we get

$$m_2\frac{d^2\vec{r}}{dt^2} = -\frac{Gm_1m_2}{r^2}\frac{\vec{r}}{\|r\|} \tag{6}$$

The equation of motion is then converted into a differential equation of the second order.

$$\frac{d^2\vec{r}}{dt^2} = -\frac{Gm_1}{r^3}\vec{r} \tag{7}$$

Equation (7) can be written as two first-order differential equations, as shown below.

$$\frac{d\vec{V}}{dt} = -\frac{Gm_1}{r^3}\vec{r} \tag{8}$$

$$\frac{d\vec{r}}{dt} = \vec{V} \tag{9}$$

Two systems of first-order differential equation and initial condition are formulated as follows.

$$x\text{-axis} \qquad \frac{dV_x}{dt} = -\frac{Gm_1}{r^3}r_x$$

$$\frac{dr_x}{dt} = V_x \tag{10}$$

$$y\text{-axis} \qquad \frac{dV_y}{dt} = -\frac{Gm_1}{r^3}r_y$$

$$\frac{dr_y}{dt} = V_y \tag{11}$$

Where
$G = 6.6743 \times 10^{-11}\ m^3\,kg^{-1}\,s^{-2}$
$m_1 = 1.989 \times 10^{30} kg$
$r = 1.496 \times 10^{11} m$
$v = 29780\ ms^{-1}$

2.2 Euler's Method

Ordinary differential equations (ODEs) can be solved numerically using Euler's method. The method assumes that a constant slope that is established by the derivative at the present position may be used to approximate the function $y(x)$. Equations (10) and (11), which describe the motion equations, were solved in this work using Euler's method. Using a time step of h, the following formulae were utilized to iteratively generate the succeeding position (x_{i+1}, y_{i+1}) and velocity components $(v_{x,i+1} v_{y,i+1})$.

$$x_{i+1} = x_i + v_{x,i}\, h \tag{12}$$

$$y_{i+1} = y_i + v_{y,i}\, h \tag{13}$$

The acceleration components were calculated based on the gravitational force equations:

$$a_{x,i} = -\frac{Gm_1}{r_i^3} x_i \tag{14}$$

$$a_{y,i} = -\frac{Gm_1}{r_i^3} y_i \tag{15}$$

2.3 Fourth Order Runge-Kutta Method

Ordinary differential equations (ODEs) can also be solved numerically with the Fourth Order Runge-Kutta (RK4) method. Assume that h is the distance between the starting location (x_i, y_i) and the subsequent new point (x_{i+1}, y_{i+1}). The following new point $((x_{i+1}, y_{i+1})$ can be computed using RK4 method by

$$y_{i+1} = y_i + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4) \tag{16}$$

where the slopes are

$$\begin{aligned} K_1 &= f(x_i, y_i) \\ K_2 &= f(x_i + \tfrac{1}{2}h, y_i + K_1 h) \\ K_3 &= f(x_i + \tfrac{1}{2}h, y_i + K_2 h) \\ K_4 &= f(x_i + h, y_i + K_3 h) \end{aligned} \tag{17}$$

K_1 denotes the slope at the beginning of the interval. K_2 denotes the slope at the midpoint of the interval by using K_1 to estimate it. K_3 denotes the slope at the midpoint of the interval by using K_2 to estimate it. K_4 denotes the slope at the end of the interval by using K_3 to estimate it.

Then, two systems of first-order ODE which are equation (10) and equation (11) will be solved by RK4 method. Now, the slope of all equations is calculated as below. For the x-axis, they are

$$\begin{aligned} K_{vx,1} &= f(r_{x,1}) = -\frac{Gm_1}{r^3} r_{x,i} \\ K_{rx,1} &= f(v_{x,i}) = v_{x,i} \\ K_{vx,2} &= f_1\left(r_{x,i} + \tfrac{1}{2}K_{rx,1}h\right) = -\frac{Gm_1}{r^3}(r_{x,i} + \tfrac{1}{2}K_{rx,1}h) \\ K_{rx,2} &= f_2\left(v_{x,i} + \tfrac{1}{2}K_{vx,1}h\right) = v_{x,i} + \tfrac{1}{2}K_{vx,1}h \\ K_{vx,3} &= f_1\left(r_{x,i} + \tfrac{1}{2}K_{rx,2}h\right) = -\frac{Gm_1}{r^3}(r_{x,i} + \tfrac{1}{2}K_{rx,2}h) \\ K_{rx,3} &= f_2\left(v_{x,i} + \tfrac{1}{2}K_{vx,2}h\right) = v_{x,i} + \tfrac{1}{2}K_{vx,2}h \\ K_{vx,4} &= f_1\left(r_{x,i} + \tfrac{1}{2}K_{rx,3}h\right) = -\frac{Gm_1}{r^3}(r_{x,i} + \tfrac{1}{2}K_{rx,3}h) \\ K_{rx,4} &= f_2\left(v_{x,i} + \tfrac{1}{2}K_{vx,3}h\right) = v_{x,i} + \tfrac{1}{2}K_{vx,3}h \end{aligned} \tag{18}$$

Then, the value of the next new point is considered by

$$v_{x,i+1} = v_{x,i} + \tfrac{1}{6}\left(K_{vx,1} + 2K_{vx,2} + 2K_{vx,3} + K_{vx,4}\right)h \tag{19}$$

$$r_{x,i+1} = r_{x,i} + \tfrac{1}{6}\left(K_{vx,1} + 2K_{vx,2} + 2K_{vx,3} + K_{vx,4}\right)h \tag{20}$$

The same procedure is applied to the y-axis.

2.4 Convergence Test

In the context of numerical approximation techniques, convergence tests are essential for evaluating the correctness and dependability of the answer. This test helps assess whether the correct solution is being processed by the numerical approximation method.

Since this study lacks a precise solution to serve as a guide for the approximation solution, a convergence test will be performed to ascertain the approximation solution's reliability. Euler's method and RK4 method will be used to conduct the convergence test.

The error for a given time step h is calculated by comparing the solutions at two successive step sizes h_i and h_{i+1}. The formula for the error at each step is given by:

$$E(h) = \left\| r_{h_i} - r_{h_{i+1}} \right\| \tag{21}$$

where r_{h_i} and $r_{h_{i+1}}$ are solutions with step sizes h_i and h_{i+1}.

The error $E(h)$ measures the difference between the two solutions, and it helps quantify how much the solution improves as the step size decreases.

The rate of convergence of a numerical method tells us how quickly the error decreases as the step size gets smaller. This is commonly expressed using Big-O notation, which describes the relationship between the step size and the error.

For Euler's method, which is a first-order method, the error typically decreases as $O(h)$. This means that if we halve the step size h, the error will decrease approximately by a factor of 2. The accuracy improves linearly with smaller step sizes, which is expressed as:

$$E(h) = O(h) \tag{22}$$

For higher-order methods like the RK4 method, the error decreases much faster. For the RK4 method, the error decreases as $O(h^4)$, meaning that halving the step size reduces the error by about a factor of $2^4 = 16$. This indicates faster convergence and higher accuracy as the step size becomes smaller. For RK4 method, it is expressed as:

$$E(h) = O(h^4) \tag{23}$$

The convergence rate can be determined by examining the ratio of errors for two successive step sizes. Given two consecutive step sizes h_i and h_{i+1}, the convergence rate p is computed as:

$$p = \frac{\log\left(\frac{E(h_i)}{E(h_{i+1})}\right)}{\log\left(\frac{h_i}{h_{i+1}}\right)} \tag{24}$$

Where $E(h_i)$ and $E(h_{i+1})$ are the errors at a step size h_i and h_{i+1}.

As an alternative way, the relationship between the error and the step size can be analysed using logarithmic scaling. Both the step sizes h and the corresponding errors $E(h)$ were transformed into their logarithmic values. Using MATLAB's polyfit function, a straight line was fitted to this data, obtaining the slope, which represents the convergence rate, p. The slope shows how quickly the error decreases as the step size becomes smaller.

The convergence rate p tells us how the error changes in relation to the change in step size. A higher value of p indicates faster convergence.

3. Result and Discussion

The comparison of Earth's orbital computations using Euler's method and the Fourth Order Runge-Kutta (RK4) method is shown in this section. The accuracy and convergence behaviour of the results is assessed.

3.1 Euler's Method Result

Fig. 2 shows the elliptical orbit of Earth simulated using Euler's method. This graph represents the numerical approximation of Earth's trajectory around the Sun.

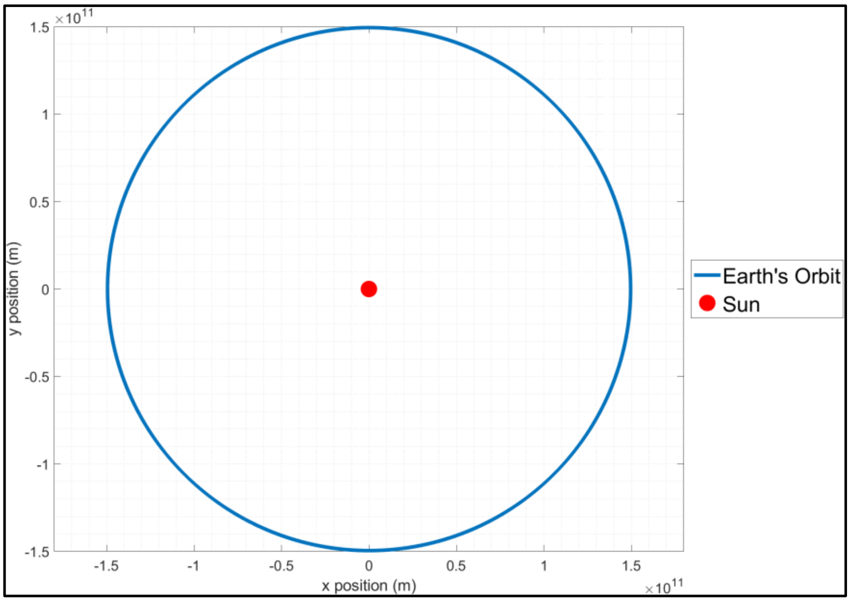

Figure 2 *Earth's Orbit by Euler's Method*

The Earth's orbit calculated using Euler's method shows an elliptical trajectory, as supported by the values in Table 1. The table presents the longest vertical and horizontal distances calculated using Euler's method. These values provide numerical evidence of the elliptical nature of Earth's orbit. These distances reflect the slight differences between the major and minor axes, consistent with the low eccentricity of Earth's orbit.

Table 1 Longest Distance Between Two Earths by Euler's Method

Direction	Longest Distance (m) (10^{11})
Vertically	2.9902138714561
Horizontally	2.9902141825845

The slight differences between the vertical and horizontal distances confirm that Earth's orbit is an ellipse. However, these values also reflect the limitations of Euler's method. The accumulated numerical errors can slightly distort the orbit's shape, especially over longer simulations. This highlights the method's dependence on step size for accuracy. While Euler's method can approximate the orbit effectively, its linear error reduction rate limits its precision, especially in cases requiring high accuracy.

The results in Table 1 demonstrate Euler's method's ability to approximate the orbit but also reveal its weaknesses. The error accumulation over time can lead to deviations, particularly for long-term simulations. This makes Euler's method less suitable for applications like satellite trajectory planning, where precision is critical.

Fig. 3 illustrates the convergence test for Euler's method, highlighting the relationship between step size and error. The step sizes used were 80, 60, 40, 20, and 10 seconds.

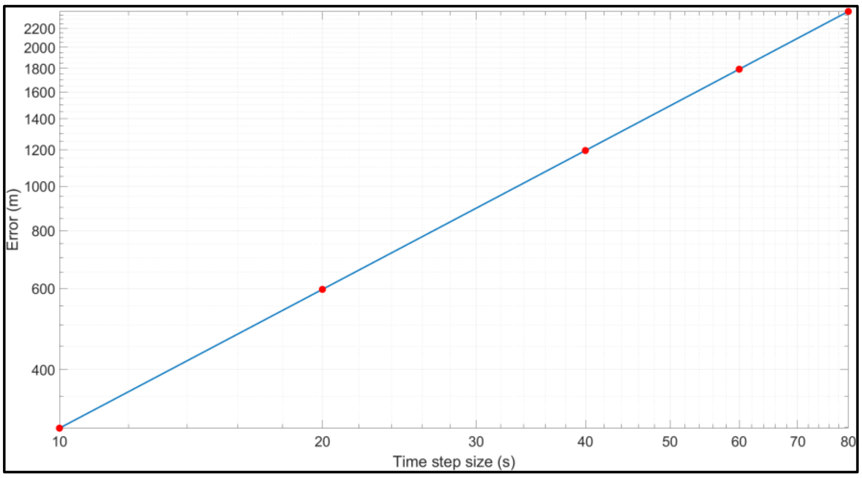

Figure 3 Convergence Test for Euler's Method

The convergence rate of 1.0012 confirms the first-order accuracy of Euler's method. The linear relationship between step size and error indicates that reducing the step size proportionally reduces the error. However, this slow rate of error reduction means that achieving high precision requires very small step sizes, which increases computational demands. The convergence test results in Fig. 3 underline the trade-offs involved in using Euler's method. While it is simple to implement, its linear error reduction rate makes it inefficient for high-precision simulations. This limitation highlights the need for higher-order methods like the RK4 method for more accurate results.

3.2 Fourth Order Runge-Kutta Method

Fig. 4 depicts the Earth's orbit simulated using the Fourth Order Runge-Kutta (RK4) method, showing the method's ability to accurately represent the elliptical trajectory.

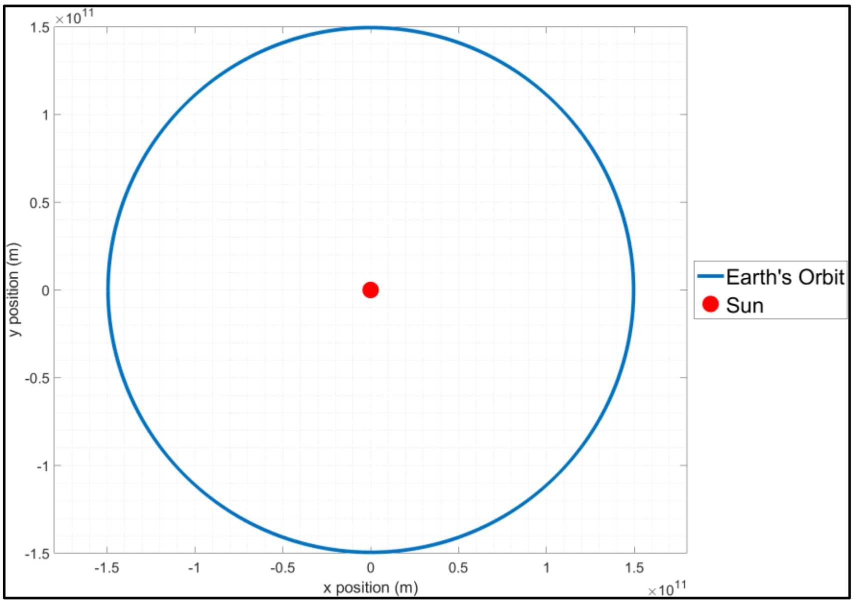

Figure 4 Earth's Orbit by RK4 Method

The RK4 method simulates the elliptical shape of Earth's orbit, as evidenced by the values in Table 2 which shows the longest vertical and horizontal distances of Earth's orbit calculated using the RK4 method. These values validate the elliptical nature of the orbit and align with the physical characteristics of Earth's motion around the Sun. The visualisation also proves, that Earth's orbit has a low eccentricity.

Table 2 Longest Distance Between Two Earths by RK4 Method

Direction	Longest Distance (m) (10^{11})
Vertically	2.9902131025811
Horizontally	2.9902136063996

The minimal differences between the vertical and horizontal distances demonstrate the RK4 method's superior accuracy in simulating the elliptical orbit. Unlike Euler's method, the RK4 method effectively minimizes numerical errors, even over long-term simulations. These results highlight the RK4 method's ability to handle complex orbital dynamics with precision.

Fig. 5 illustrates the convergence test for the RK4 method, demonstrating its fourth-order accuracy. The step sizes used were 2 678 400, 604 800, 86 400, and 3 600 seconds which respectively equal to one month, one week, one day and one hour.

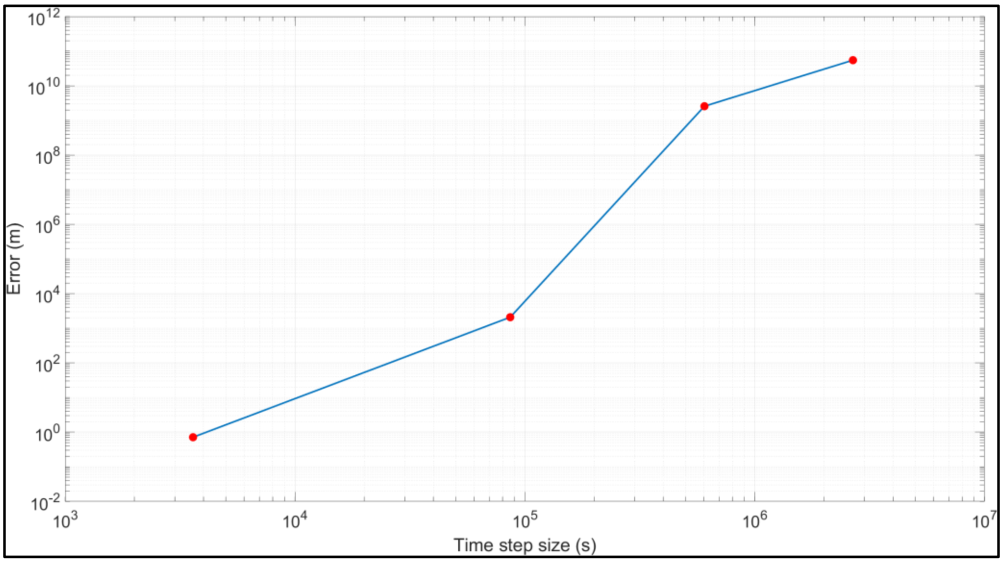

Figure 5 Convergence test for RK4 Method

The convergence rate of 4.0296 confirms the RK4 method's ability to rapidly reduce errors with smaller step sizes. This result highlights the method's efficiency in achieving high precision, as the error decreases exponentially compared to Euler's method. The convergence test in Fig. 5 emphasizes the advantages of higher-order methods like RK4. Its rapid error reduction rate ensures reliable results with fewer steps, making it an essential tool for high-fidelity simulations in celestial mechanics.

4. Conclusion

In this study, we perform a convergence test to evaluate the accuracy of Euler's method and Fourth Order Runge-Kutta (RK4) method by calculating the error at progressively smaller step sizes. By comparing the solutions at different step sizes, we can assess how the error decreases as the step size is refined. The error for Euler's method decreases linearly, as $O(h)$, while the error for RK4 method decreases more rapidly, as $O(h^4)$. This shows that higher-order methods converge faster, providing more accurate results for smaller step sizes. Euler's method, while simple to implement, is less efficient for high-precision applications due to its slower convergence. In this study, both Euler's method and RK4 method produced nearly identical orbits within the chosen simulation time frame. However, the convergence test showed that Euler's method, which exhibits first-order convergence $O(h)$ accumulates errors more rapidly with decreasing step size. In contrast, the RK4 method with its fourth-order convergence $O(h^4)$ demonstrated a much faster reduction in error as the step size decreased. Although the visual differences between the two methods were minimal for the selected time span, the convergence test indicates that RK4 method would provide significantly more accurate results with smaller step sizes or over longer simulations. Thus, while both methods performed similarly within the given timeframe, the RK4 method's superior convergence rate makes it more suitable for applications requiring higher precision over extended periods.

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Conflict of Interest

Authors declare that there is no conflict of interests regarding the publication of the paper.

Author Contribution

The authors confirm contribution to the paper as follows: ***study conception and design:*** *Daniel Aiman Oon Jeffrey Oon;* ***solve the equation:*** *Daniel Aiman Oon Jeffrey Oon;* ***analysis and interpretation of results:*** *Daniel Aiman Oon Jeffrey Oon, Mahathir Mohamad;* ***draft manuscript preparation:*** *Daniel Aiman Oon Jeffrey Oon, Mahathir Mohamad. All authors reviewed the results and approved the final version of the manuscript.*

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