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# Differential Transform Method for Solving Linear Ordinary Differential Equations with Constant Coefficients

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**Abstract**

In this research, a semi-analytical method called the differential transform method (DTM) was used to solve linear ordinary differential equations (ODEs). DTM is a powerful mathematical technique that transforms differential equations into algebraic recurrence relations, making them easier to solve. This method is particularly useful in obtaining approximate analytical solutions without requiring complex computational techniques. With the help of MATLAB R2024a, approximate solutions for linear ODEs were obtained, along with graph and absolute error calculations. The numerical results obtained from DTM were compared with the exact solutions derived using the Elzaki transform, which is an integral transform method know for providing exact analytical solutions to differential equations. The comparison aimed to evaluate the reliability of DTM in solving linear ODEs. This research found that the accuracy of DTM depends on several factors, including the type of differential equation being solved and the number of terms considered in the DTM series expansion. Increasing the number of terms in the series improves the accuracy of the approximate solution, reducing the error when compared to the exact solution. DTM successfully solved two examples of linear ODEs. The results indicated that DTM requires lower computational effort than traditional numerical methods while still providing accurate approximations. This makes DTM one of the simplest methods to apply and can solve differential problems.

## 1. Introduction

Differential equations have long been fundamental in various fields of applied mathematics, particularly in modelling real-world phenomena across engineering, physics, and computational sciences. The challenge of solving these equations has led to the development of numerous analytical and numerical methods. Among these, the DTM and Elzaki transform have been widely explored due to their effectiveness in solving linear and nonlinear ODEs with constant or variable coefficients. The DTM, first introduced by [1], provides an alternative approach to solving differential equations by converting them into recurrence relations, allowing solutions to be expressed in the form of a Taylor series. This method has been extensively applied to solve ODEs, partial differential equations (PDEs), and integro-differential equations [2],[3],[4]. The DTM is particularly advantageous due to its ability to solve nonlinear differential problems directly, making it a widely used tool in mathematical and engineering applications [5],[6]. However, the accuracy of the DTM depends on the number of terms considered in the series expansion, with higher-order terms improving precision but also increasing computational effort [7].

On the other hand, the Elzaki transform, introduced by [8], is a relatively newer integral transform method derived from the classical Fourier integral. It has been recognized for its mathematical simplicity and capability to handle a wide range of ODEs and PDEs effectively [9]. The Elzaki transform has been applied in solving differential equations with variable coefficients and systems of ODEs, demonstrating its effectiveness in obtaining exact analytical solutions [10],[11]. Studies have also explored the relationship between the Elzaki and Laplace transforms, highlighting its efficiency in solving linear ODEs [12]. Although the Elzaki transform is powerful, it remains underexplored compared to more established methods such as Laplace and Sumudu transforms, limiting its application in broader research areas [13],[14].

Despite the advantages of both methods, there is a need to assess the accuracy and computational efficiency of the DTM compared to exact solutions obtained using the Elzaki transform. DTM provides approximate solutions based on Taylor series expansion, the Elzaki transform offers exact solutions, making it a suitable benchmark for evaluating the performance of DTM [15]. Previous studies have explored the applicability of DTM in solving various types of differential equations, but limited research has focused on comparing its accuracy with exact solutions obtained through integral transforms. Addressing the gap, this study aims to solve linear ODEs using the DTM and compare the result with the exact solutions obtained from the Elzaki transform. By analysing the absolute error between these methods, this research seeks to determine the extent to which DTM can approximate exact solutions and assess the trade-off between accuracy and computational complexity. The findings of this study will contribute to a better understanding of the applicability and efficiency of DTM in solving linear ODEs, providing insights into its potential for broader applications in mathematical modelling and engineering analysis.

The objectives of this research are to explore the application of DTM for solving linear ODEs with constant coefficients, obtains solutions using this method, and compare its accuracy with the exact solutions derived from Elzaki transform. This study aims to highlight the strengths and limitations of the DTM while evaluating its efficiency as an alternative semi-analytical technique for solving differential equations.

## 2. Research Methodology

To find an adequate and reasonable solution while solving linear ODEs with constant coefficients by comparing DTM and exact solution by Elzaki transform, it is important for us to use and apply precise and accurate theorem and techniques.

### 2.1 Ordinary Differential Equation

A differential equation is an equation in which the function that is the unknown may appear in the equation together with its derivatives [16]. The general equation for ODE is defined as in equation (1) [17]:

$$y' = \frac{dy}{dx} = f(x,y) \tag{1}$$

A linear ODE of $n$ with constant coefficients can be expressed in the general form of equation (2),

$$a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1 \frac{dy}{dx} + a_0 y = g(x) \tag{2}$$

where $y = y(x)$ is the dependent variable, $\frac{d^n y}{dx^n}$ represent the $n$-th derivative of $y$ respect to $x$, $a_0, a_1, ..., a_n$ are constant coefficients, and $g(x)$ is a function of $x$ that represents the non-homogeneous term. If $g(x) = 0$, the formula becomes in form of equation (3),

$$a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1 \frac{dy}{dx} + a_0 y = 0 \tag{3}$$

which is referred to as a homogenous linear ODE.

## 2.2 Differential Transform Method

The DTM of a function $y(x)$ for the $k$th derivatives is defined as in equation (4), if the function $y(x)$ is continuously differentiable,

$$Y(k)=\frac{1}{k!}\left[\left.\frac{d^k y(x)}{dx^k}\right|_{x=0}\right] \quad (4)$$

where $Y(k)$ is a transformed function and $y(x)$ is the original function. The differential inverse transform is defined as in equation (5)

$$y(x)=\sum_{k=0}^{\infty}Y(k)(x-x_0)^k . \quad (5)$$

Table 1 lists the fundamental mathematical operations performed by DTM.

**Table 1** The fundamental mathematical operations by [10]

| Original function | Transformed function |
|---|---|
| $y(x)=u(x)\pm v(x)$ | $Y(k)=U(k)\pm V(k)$ |
| $y(x)=cg(x)$ | $Y(k)=cG(k)$, $c$ is constant |
| $y(x)=y'(x)$ | $Y(k)=(k+1)Y(k+1)$ |
| $y(x)=y''(x)$ | $Y(k)=(k+1)(k+2)Y(k+2)$ |
| $y(x)=\frac{\partial^m}{\partial x^m}y(x)$ | $Y(k)=(k+1)(K+2)...(k+m)Y(k+m)$ |
| $y(x)=u(x)v(x)$ | $Y(k)=\sum_{m=0}^{k}U(m)V(k-m)$ |
| $y(x)=x^n$ | $Y(k)=\delta(k-n)=\begin{cases}1, & k=n\\ 0, & k\neq n\end{cases}$ |

## 2.3 Elzaki Transform

The Elzaki transform denoted by the operator $E$ defined by the integral equations as in equation (6).

$$E\left[y(x)\right]=T(v)=v\int_0^{\infty}y(x)e^{\frac{-x}{v}}dx,\ \ x\geq 0,\ \ k_1\leq v\leq k_2 \quad (6)$$

Table 2 lists the theorem of Elzaki transform of the function $y(x)$ given by $E\left[y(x)\right]=T(v)$.

**Table 2** Theorem of Elzaki transform [5]

| Original function | Transformed function |
|---|---|
| $y(x)$ | $T(v)$ |
| $y'(x)$ | $\frac{T(v)}{v}-vy(0)$ |
| $y''(x)$ | $\frac{T(v)}{v^2}-y(0)-vy'(0)$ |
| $y^{(n)}(x)$ | $\frac{T(v)}{v^n}-\sum_{k=0}^{n-1}v^{2-n+k}y^{(k)}(0)$ |

Next, Table 3 shows the inverse Elzaki transform of some functions, which required in this research.

**Table 3** Inverse Elzaki transform of some functions [7]

| $y(x)$ | $E[y(x)]=T(v)$ |
|---|---|
| 1 | $v^2$ |
| $x$ | $v^3$ |
| $e^{ax}$ | $\frac{v^2}{1-av}$ |

## 3. Result

**Example 1.** Consider the following linear ODE:

$$y''-3y+2y=0 \tag{7}$$

with initial conditions

$$y(0)=1, y'(0)=4$$

and given the exact solution [8]

$$e^x(3e^x-2).$$

By using the theorem of DTM from Table 1 to the equation (7), we obtain equation (8),

$$(k+1)(k+2)Y(k+2)-3(k+1)Y(k+1)+2Y(k)=0$$

$$Y(k+2)=\frac{3(k+1)Y(k+1)-2Y(k)}{(k+1)(k+2)} \tag{8}$$

with initial conditions

$$Y(0)=1 \text{ and } Y(1)=4. \tag{9}$$

Then, we substitute $k=0,\ Y(0)=1,\ Y(1)=4,$

$$\begin{aligned} Y(0+2) &= \frac{3(0+1)Y(0+1)-2Y(0)}{(0+1)(0+2)} \\ &= \frac{3Y(1)-2Y(0)}{2} \\ Y(2) &= 5 \end{aligned} \tag{10}$$

The recurrence relation equation (7) at $k=1,2,3,4,5,6,7$, then, we will obtain the following values.

$$Y(3)=\frac{11}{3}$$

$$Y(4)=\frac{23}{12}$$

$$Y(5)=\frac{47}{60}$$
$$Y(6)=\frac{19}{72}$$
$$Y(7)=\frac{191}{2520}$$
$$Y(8)=\frac{383}{20160}$$
$$Y(9)=\frac{767}{181440}$$

Therefore, combine all the term and do the Taylor series until 10th term.

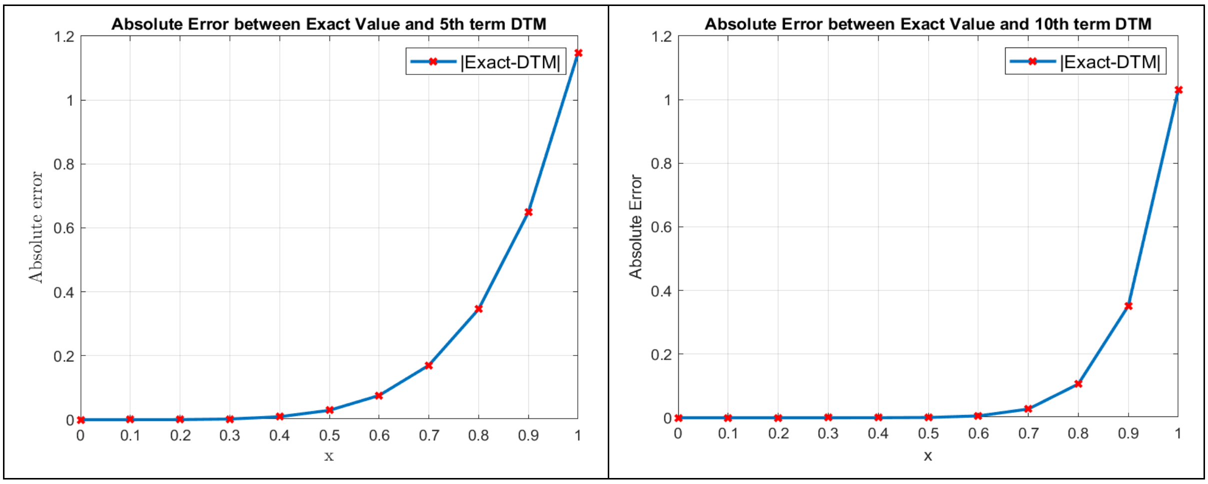
$$Y(x)=1+4x+5x^2+\frac{11x^3}{3}+\frac{23x^4}{12}+\frac{47x^5}{60}+\frac{19x^6}{72}+\frac{191x^7}{2520}+\frac{383x^8}{20160}+\frac{767x^9}{181440} \quad (11)$$

Using MATLAB R2024a, the numerical solution for Eq. (7) is obtained, including numerical solution and graph of absolute error. Table 4 and Fig. 1 show the overall results.

**Table 4** Numerical solution for Example 1

| $x$ | Exact value | DTM | | Absolute Error \|Exact Value - DTM\| | |
|---|---|---|---|---|---|
| | | 5th term | 10th term | 5th term | 10th term |
| 0.0 | 1.000000E+00 | 1.000000E+00 | 1.000000E+00 | 0.000000E+00 | 0.000000E+00 |
| 0.1 | 1.453866E+00 | 1.453858E+00 | 1.453866E+00 | 8.105000E-06 | 0.000000E+00 |
| 0.2 | 2.032669E+00 | 2.032400E+00 | 2.032669E+00 | 2.685766E-04 | 1.000000E-10 |
| 0.3 | 2.766639E+00 | 2.764525E+00 | 2.766639E+00 | 2.113786E-03 | 5.300000E-09 |
| 0.4 | 3.692973E+00 | 3.683733E+00 | 3.692973E+00 | 9.240057E-03 | 9.560000E-08 |
| 0.5 | 4.857403E+00 | 4.828125E+00 | 4.857402E+00 | 2.927794E-02 | 9.081000E-07 |
| 0.6 | 6.316113E+00 | 6.240400E+00 | 6.316107E+00 | 7.571317E-02 | 5.735200E-06 |
| 0.7 | 8.138094E+00 | 7.967858E+00 | 8.138067E+00 | 1.702362E-01 | 2.733770E-05 |
| 0.8 | 1.040802E+01 | 1.006240E+01 | 1.040791E+01 | 3.456154E-01 | 1.060642E-04 |
| 0.9 | 1.322974E+01 | 1.258053E+01 | 1.322938E+01 | 6.492112E-01 | 3.516650E-04 |
| 1.0 | 1.673060E+01 | 1.558333E+01 | 1.672957E+01 | 1.147271E+00 | 1.030125E-03 |

Table 4 presents a comparison between exact values and the numerical solutions obtained using the DTM at the 5th term and 10th term approximations. Alongside these approximations, Table 4 also provides the absolute errors, which reflect the difference between the exact values and their corresponding DTM approximations. As shown in the table, the error decreases as more terms are included in the series, with the 10th term approximation generally providing a more accurate result than the 5th term.




(a) (b)

**Fig. 1** Absolute error graph of (a) 5th term; (b) 10th term

Fig. 1(a) illustrates the absolute error of the 5th term DTM, meanwhile Fig. 1(b) presents the absolute error of the 10th term DTM. Obviously that the 5th term DTM exhibits higher absolute errors compared to the 10th term DTM, particularly as $x$ increases. This indicates that the 5th term approximation is less accurate in representing the exact solution. In contrast, the 10th term DTM provides a significantly lower absolute error, demonstrating that increasing the number of terms improves the accuracy of the numerical approximation.

**Example 2.** Consider the following linear ODE as in equation (12),

$$y'+2y=x \tag{12}$$

with initial condition

$$y(0)=1$$

and give the exact solution (Elzaki, 2011)

$$\frac{1}{2}x+\frac{5}{4}e^{-2x}-\frac{1}{4}.$$

By using the theorem of DTM from Table 1 to the equation (12), we obtain

$$\begin{aligned}(k+1)Y(k+1)+2Y(k)&=\delta(k-n)\\ Y(k+1)&=\frac{\delta(k-n)-2Y(k)}{(k+1)}\end{aligned} \tag{13}$$

with initial condition

$$Y(0)=1 \tag{14}$$

Substitute $k=0,\ Y(0)=1$,

$$\begin{aligned}Y(0+1)&=\frac{\delta(0-1)-2Y(0)}{(0+1)}\\ Y(1)&=-2\end{aligned} \tag{15}$$

The recurrence relation equation (12) at $k=1,2,3,4,5,6,7,8$, we obtain the following values,

$$Y(2)=\frac{5}{2}$$
$$Y(3)=-\frac{5}{3}$$
$$Y(4)=\frac{5}{6}$$
$$Y(5)=-\frac{1}{3}$$
$$Y(6)=\frac{1}{9}$$
$$Y(7)=-\frac{2}{63}$$
$$Y(8)=\frac{1}{126}$$
$$Y(9)=-\frac{1}{567}$$

Therefore, combine all the term and do the Taylor series until $10^{th}$ term.

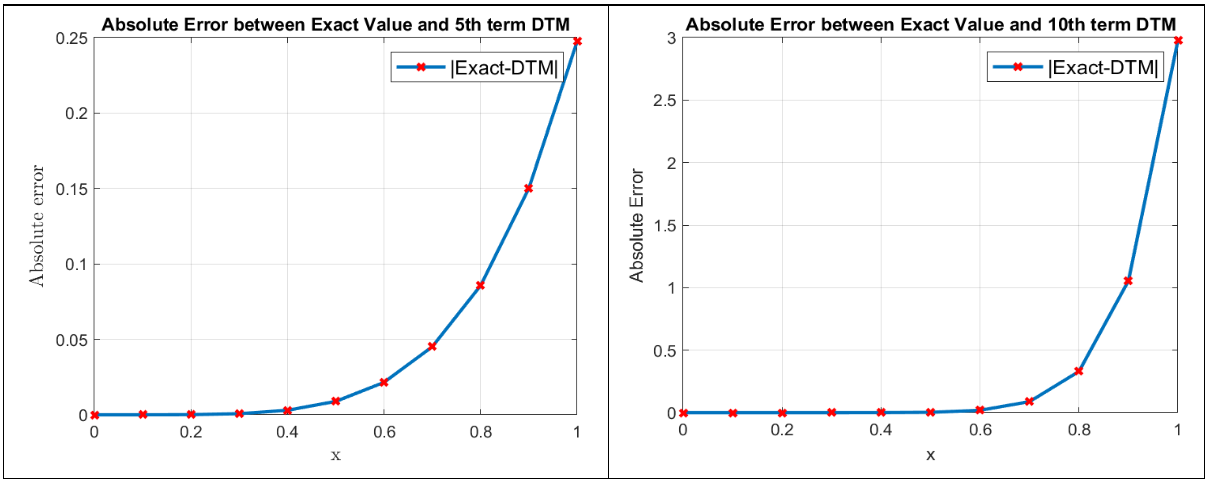
$$Y(x)=1-2x+\frac{5x^2}{2}-\frac{5x^3}{3}+\frac{5x^4}{6}-\frac{x^5}{3}+\frac{x^6}{9}-\frac{2x^7}{63}+\frac{x^8}{126}-\frac{x^9}{567} \quad (16)$$

Using MATLAB R2024a, the numerical solution for equation (12) is obtained, including numerical solution and graph of absolute error. Table 5 and Fig. 2 show the overall results.

**Table 5** Numerical solution for Example 2

| $x$ | Exact value | DTM | | Absolute Error \|Exact Value - DTM\| | |
|---|---|---|---|---|---|
| | | 5th term | 10th term | 5th term | 10th term |
| 0.0 | 1.000000E+00 | 1.000000E+00 | 1.000000E+00 | 0.000000E+00 | 0.000000E+00 |
| 0.1 | 8.234134E-01 | 8.234167E-01 | 8.234134E-01 | 3.225300E-06 | 0.000000E+00 |
| 0.2 | 6.879001E-01 | 6.880000E-01 | 6.879001E-01 | 9.994250E-05 | 0.000000E+00 |
| 0.3 | 5.860145E-01 | 5.867500E-01 | 5.860145E-01 | 7.354549E-04 | 2.000000E-09 |
| 0.4 | 5.116612E-01 | 5.146667E-01 | 5.116612E-01 | 3.005462E-03 | 3.450000E-08 |
| 0.5 | 4.598493E-01 | 4.687500E-01 | 4.598490E-01 | 8.900699E-03 | 3.156000E-07 |
| 0.6 | 4.264928E-01 | 4.480000E-01 | 4.264908E-01 | 2.150724E-02 | 1.921500E-06 |
| 0.7 | 4.082462E-01 | 4.082462E-01 | 4.082374E-01 | 4.517046E-02 | 8.829200E-06 |
| 0.8 | 4.023706E-01 | 4.534167E-01 | 4.023376E-01 | 8.562935E-02 | 3.301890E-05 |
| 0.9 | 4.066236E-01 | 4.880000E-01 | 4.065181E-01 | 1.501264E-01 | 1.055137E-04 |
| 1.0 | 4.191691E-01 | 6.666667E-01 | 4.188713E-01 | 2.474976E-01 | 2.978518E-04 |

Table 5 presents a comparison between exact values and the numerical solutions obtained using the DTM at the $5^{th}$ term and $10^{th}$ term approximations. Alongside these approximations, the Table 5 also provides the absolute errors, which reflect the difference between the exact values and their corresponding DTM approximations. Here, the error decreases as more terms are included in the series, with the $10^{th}$ term approximation generally providing a more accurate result than the $5^{th}$ term.




(a) (b)

**Fig. 2** Absolute error graph of (a) 5th term; (b) 10th term

Fig. 2(a) illustrates the absolute error of the $5^{th}$ term DTM, meanwhile Fig. 2(b) presents the absolute error of the $10^{th}$ term DTM. Here, there is an evident that the $5^{th}$ term DTM exhibits higher absolute errors compared to the $10^{th}$ term DTM, particularly as $x$ increases. This indicates that the $5^{th}$ term approximation is less accurate in representing the exact solution. In contrast, the $10^{th}$ term DTM provides a significantly lower absolute error, demonstrating that increasing the number of terms improves the accuracy of the numerical approximation.

## 4. Discussion

In this study, a computational approach was used to solve linear ODEs using the DTM. The method involved transforming the ODE into a recurrence relation and obtaining an approximate solution through a series expansion. The accuracy of DTM was analysed by comparing it with the exact solution from the Elzaki transform, where higher terms in the series led to reduced absolute error. MATLAB R2024a was used to implement DTM, perform symbolic and numerical computations, and generate graphical representations of the results. The software also facilitated error analysis to assess the method's accuracy. However, DTM has limitations, particularly its reliance on the number of terms in the series expansion, which affects accuracy. It is also more effective for problems with simple boundary conditions and may be less efficient for equations with variable coefficients. Meanwhile, MATLAB provided an efficient computational environment, the accuracy of DTM depended on the choice of terms, highlighting both its potential and its limitations compared to the Elzaki transform.

## 5. Conclusions

Two examples of linear ODEs were solved using DTM. Example 1 involved a homogenous linear ODE, while Example 2 was a non-homogenous linear ODE. The numerical results obtained from DTM were compared with exact solutions derived using Elzaki transform from previous research. As an analytical method, the Elzaki transform provides exact solutions, serving as a benchmark for evaluating the accuracy of DTM. Both examples were compared using MATLAB R2024a, which facilitated numerical computations and graphical visualization.

In conclusion, this research examined the effectiveness of the DTM in solving linear ODEs by comparing its approximate solutions with the exact solutions obtained using Elzaki transform. The study considered two cases: using DTM up to $5^{th}$ term and the $10^{th}$ term. The result demonstrated that increasing the number of terms in the DTM series expansion significantly improved the accuracy of the solution by reducing the absolute error. The comparison with exact solution showed that while 5th term approximation provided a reasonable result, the 10th term approximation yielded a much closer match, confirming that DTM's accuracy depends on the number of terms used. Additionally, MATLAB R2024a was instrumental in implementing the DTM algorithm, calculating the approximate solutions, and analysing the errors effectively. Overall, this study highlights the applicability of DTM as a semi-analytical method for solving ODEs, offering an efficient and computationally simple alternative for obtaining approximate solutions.

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## Conflict of Interest

Authors declare that there is no conflict of interests regarding the publication of the paper.

## Author Contribution

*The authors confirm contribution to the paper as follows:* ***study conception and writing:*** *Nurhikmah Shaharuddin;* ***guidance and made corrections:*** *Noor Azliza Abd Latif, Norziha Che Him. All authors reviewed the results and approved the final version of the manuscript.*

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