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Solving the HIV-AIDS Transmission Mathematical Model using Second-Order Runge-Kutta Methods

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Abstract

This study focuses on the mathematical modelling of human immunodeficiency virus - acquired immunodeficiency syndrome (HIV-AIDS) dynamics using the Susceptible-Infected-Chronic-AIDS (SICA) model. The objective is to approximate the solution of HIV-AIDS transmission in a population by estimating the proportion of individuals in the subpopulations using second-order Runge-Kutta (RK2) methods specifically the midpoint and Ralston's methods. The results are then compared to the existing solution using the fourth-order Runge-Kutta (RK4) method obtained by [1], with all computations performed in MATLAB R2023a. The results show that the susceptible subpopulation decreases, HIV infected subpopulation peaks then decline, the chronic subpopulation increases, and the AIDS subpopulation decreases, with all stabilizing at equilibrium. The comparison shows that the midpoint and Ralston methods produce results nearly identical to the RK4 method, with minor discrepancies in regions of rapid transitions. Norm analysis shows Ralston's method outperforms the midpoint method, but both proved to be reliable for solving the SICA model.

1. Introduction

HIV, a virus that targets the body's immune system. It can progress to AIDS if left untreated which represents the most progressed phase of the illness [2]. HIV destroys CD4+ T-cells, a specialized class of white blood cells that help maintain health by combating diseases, leading to a weakened immune system [3]. Even after almost 40 years since its initial discovery, HIV-AIDS continues to represent one of the most significant disease burdens globally, with one of the highest mortality rates among infectious diseases. Based on its severity, the study about HIV-AIDS has been done by many researchers for predicting the spread of the disease. To understand HIV-AIDS transmission deeper, it is modelled by a system of ordinary differential equations (ODEs).

According to [4] mathematical modelling is a branch of mathematics that represents problems using mathematical expressions, resulting in a mathematical model. Once a mathematical model is constructed, mathematical analysis, often combined with computer simulations, allows us to explore the global behaviours of the model, examining the implications of our assumptions. As a result, we can predict various aspects of an epidemic, such as the number of individuals expected to be infected, the duration, the peak incidence, and the entire epidemic curve, providing the expected number of cases at each point in time [5].

[6] have done a simulation of a Susceptible-Infected-Removed (SIR) mathematical model of HIV transmission dynamics using the classical Euler's method. The R_0 of the model is computed, and it is demonstrated that the model is locally asymptotically stable under two conditions which are the removed group includes individuals who receive treatment and those who use preventive measures such as condoms. The results obtained from this

study indicate that the epidemic can be controlled within a finite period. In addition, A simulation of the dynamics of a Susceptible-Infected-AIDS (SIA) model of HIV-AIDS spread has been done by [4] using mathematical software. The authors found that as time approaches infinity, the number of individuals in the susceptible, infected, and AIDS cases sub-populations will tend to stabilize and the graph approaches a stable asymptotic equilibrium point, indicating that HIV and AIDS will persist.

[7],[8] has done a study about the local and global stability of the equilibrium points of SICA model for HIV-AIDS. It was found that the global stability of the unique endemic equilibrium was proven. Hence, it can be concluded that the disease remains consistently present in the population. Moreover, [3] developed the Grunwald Letnikov nonstandard finite difference scheme with the aim to solve a fractional epidemic model of HIV-AIDS disease transmission. Through this study, it is observed that the proposed scheme converges to the disease-free steady state and the endemic steady state. [9] solved the SIA model using fourth-order Runge-Kutta (RK4) method and discovered that the method effectively resolves HIV-AIDS transmission, with numerical solutions indicating that over extended periods which is approaching infinity, the susceptible, infected, AIDS subpopulations, and the total population will persist without extinction and will eventually stabilize, reaching their equilibrium points.

Due to the complexity of the disease dynamics and numerous factors involved such as varying transmission rates, solving the system of ODEs that model its transmission analytically is highly challenging. Consequently, researchers solved it with the help of numerical methods to tackle the complexities effectively. However, none have utilized the RK2 methods to solve the SICA model thus far specifically the midpoint and Ralston's method.

According to [10], the midpoint rule is easier to apply and generally outperforms the trapezoidal rule, whether for manual calculation or by using a machine. Moreover, [11] found that the effectiveness of the midpoint method has been demonstrated in approximating the solution of a two-point boundary value problem for a functional differential equation. [12] have conducted a comparative analysis of several methods, including Taylor's series method, Euler's method, and the first, second, third, and fourth-order Runge-Kutta methods. These methods were used in solving initial value problems (IVPs) of ODEs using different step sizes and implemented in MATLAB. The findings of this study demonstrate that the RK4 method provides the highest level of accuracy in solving ODEs, whereas Euler's method exhibits the lowest accuracy. The error ranges for Heun's method, the midpoint method, and Ralston's method lie within 0 and 0.02, indicating that these methods remain reliable for ODEs computations.

Hence, the purpose of this study is to utilize midpoint and Ralston's method to explore its potential in approximating the solution of HIV-AIDS transmission in a population by estimating the proportion of individuals in the susceptible, HIV-infected, chronic and AIDS subpopulations using RK2 methods. Since it is well known that the RK4 method is generally superior to RK2 [13], this study will compare the RK4 solution from existing research conducted by [1] with the solution obtained using RK2 method developed in this study to evaluate whether the RK2 method performs as effectively as the RK4 method.

2. Research Method

The HIV-AIDS model analysed in this study is the SICA model, initially developed by [14]. This study utilizes two RK2 methods, specifically the midpoint and Ralston's methods, to solve the SICA model using MATLAB.

2.1 SICA HIV-AIDS Model

The spread of HIV-AIDS within a human population begins with vulnerable individuals who become infected with HIV. Subsequently, when HIV is detected in a person's body, it will develop into chronic then AIDS over a certain period. Therefore, in this study, it is assumed that the human population is divided into four sub-populations that is mutually exclusive: susceptible individuals *(S),* HIV-infected individuals with no clinical symptoms or only mild ones but still capable of transmitting HIV to others *(I),* HIV-infected individuals with a low viral load (chronic stage) receiving ART treatment *(C)* and HIV-infected individuals with AIDS clinical symptoms *(A).* According to [14], the total population at time *t*, denoted by is given by

$$N(t)=S(t)+I(t)+C(t)+A(t)$$

and the transmission process of SICA epidemic model for HIV-AIDS is given by following system of ODEs

$$\begin{aligned} S'(t) &= b\,N(t)-\lambda(t)\,S(t)-\mu\,S(t) \\ I'(t) &= \lambda(t)\,S(t)-(\rho+\phi+\mu)\,I(t)+\alpha\,A(t)+\omega C(t) \\ C'(t) &= \phi I(t)-(\omega+\mu)C(t) \\ A'(t) &= \rho\,I(t)-(\alpha+\mu+d)A(t) \end{aligned} \tag{1}$$

while the effective contact rate with people infected with HIV is given by

$$\lambda(t)=\frac{\beta}{N(t)}(I(t)+\eta_C\,C(t)+\eta_A\,A(t)) \tag{2}$$

where
Λ is the recruitment rate.
μ is the natural death rate.
β is the contact rate of HIV transmission.
ϕ is the HIV treatment rate for I individuals.
ρ is the default treatment rate for I individuals.
α is the AIDS treatment rate.
ω is the default treatment rate for C individuals.
d is the death rate due to AIDS.
t is the time in years.

The modification parameter $\eta_A \geq 1$ represents the relative infectiousness of individuals with AIDS symptoms compared to those infected with HIV but without AIDS symptoms (since individuals with AIDS symptoms are more infectious than those without). On the other hand, $\eta_C \leq 1$ accounts for the partial restoration of immune function in individuals with HIV who correctly follow Antiretroviral Therapy (ART) treatment [7].

In the situation where the total population size $N(t)$ is inconsistent, it is often convenient to consider the proportions of each compartment of individuals in the population to be

$$s = \frac{S}{N}, i = \frac{I}{N}, c = \frac{C}{N}, r = \frac{R}{N}$$

Hence, by considering Equation (2), the state variables *s, i, c* and *a* satisfy the following system of ODEs:

$$\left.\begin{aligned} s'(t) &= b(1-s(t)) - \beta(i(t)+\eta_C c(t)+\eta_A a(t))s(t) - d a(t)s(t) \\ i'(t) &= \beta(i(t)+\eta_C c(t)+\eta_A a(t))s(t) - (\rho+\phi+b)i(t) + \alpha a(t) + \omega c(t) + d a(t) i(t) \\ c'(t) &= \phi i(t) - (\omega+b)c(t) + d a(t) c(t) \\ a'(t) &= \rho i(t) - (\alpha+b+d)a(t) + d a^2(t) \end{aligned}\right\} \tag{3}$$

with $s(t) + i(t) + c(t) + a(t) = 1$ for all $t \in [0, T]$. Fig. 1 shows the epidemiology scheme of Equation (1).

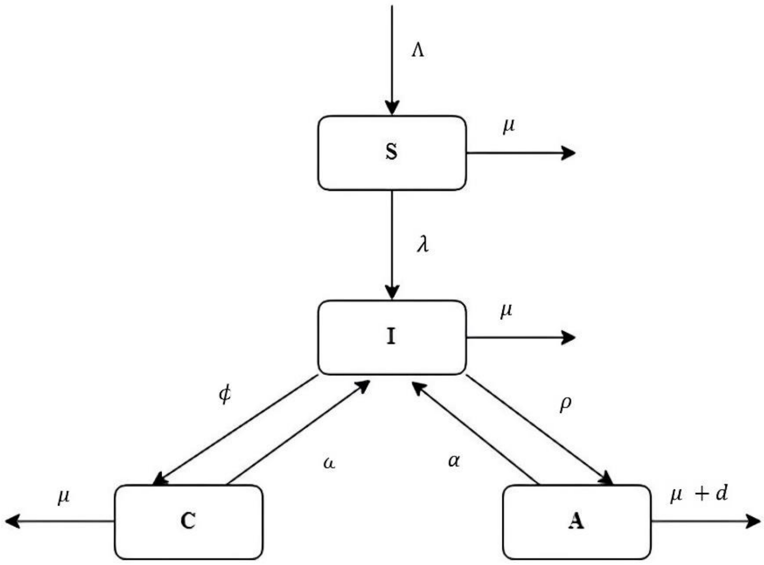

Fig. 1 Epidemiology scheme of the SICA mathematical model of HIV-AIDS [7]

To solve the system of Equation (3) using the RK2 methods, initial conditions and parameters are required. Equation (3) is to be subjected to the initial conditions given by

$$s(0) = 0.6, \qquad i(0) = 0.2, \qquad c(0) = 0.1, \qquad a(0) = 0.1$$

provided by [1] and by the final time value of T = 50 (years). Table 1 shows the fixed parameter values.

Table 1 Parameter values of the HIV-AIDS model in Equation (1)[8]

Symbol	Description	Value
μ	Natural death rate	1/69.54
b	Recruitment rate	2.1μ
β	HIV transmission rate	1.6
η_C	Modification parameter	0.015
η_A	Modification parameter	1.3
ϕ	HIV treatment rate for I individuals	1
ρ	Default treatment rate for I individuals	0.1
α	AIDS treatment rate	0.33
ω	Default treatment rate for C individuals	0.09
d	AIDS induced death rate	1

2.2 Second-order Runge Kutta methods

Runge-Kutta methods were developed around 1900 by the German mathematicians Carl David Tolme Runge and Martin Wilhelm Kutta and were named after them [15]. Runge-Kutta methods are a family of iterative techniques which can be used for approximating solutions of ODEs. The first order method or also known as Euler method uses a straightforward approach to estimate the solution. The second order method improves accuracy by incorporating an intermediate step. midpoint and Ralston's method were utilized to solve the system of Equation (2). The formula for the midpoint method is

$$y_{i+1} = y_i + hk_2\,,\ i = 0,1,2... \tag{4}$$

where

$$\begin{aligned} k_1 &= f\left(x_i, y_i\right) \\ k_2 &= f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_1\right) \end{aligned} \tag{5}$$

The formula for the Ralston's method is

$$y_{i+1} = y_i + \frac{h}{3}\left(k_1 + 2k_2\right),\ i = 0,1,2... \tag{6}$$

where

$$\begin{aligned} k_1 &= f\left(x_i, y_i\right) \\ k_2 &= f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}hk_1\right) \end{aligned} \tag{7}$$

where f is an unknown function and h is a step size.

2.3 Norm

Error analysis using vector norms was conducted to quantify the deviation between the methods. This approach provided a comprehensive assessment of the performance of the midpoint and Ralston's methods relative to the RK4 method. Specifically, the 2-Norm was employed to evaluate the differences or similarities between the graphs, capturing how far apart they are in terms of edge weights or structural properties. Additionally, the 2-Norm's sensitivity to large deviations, due to the squaring of differences, makes it particularly effective for highlighting significant structural discrepancies between graphs. The formula for the 2-Norm is

$$2\text{–}Norm = \sqrt{h\sum_{i=1}^{M}\left(y_{Method\,1}(t_i) - y_{Method\,2}(t_i)\right)^2} \tag{8}$$

3. Results and Discussion

This section demonstrates the solution of the system of nonlinear ODEs modelling the transmission of HIV-AIDS using the SICA model. The model was solved numerically using the midpoint method and Ralston's method. All calculations were performed using MATLAB software, and the resulting graphs are discussed in this section. The obtained solution was then compared to the existing solution of RK4 method provided by [1].

3.1 Midpoint Method and Ralston's Method

The midpoint method, defined in Equations (4) and (5), and Ralston's method, defined in Equations (6) and (7), were applied to obtain numerical results for the SICA HIV-AIDS model presented in Equation (3). The performance of the midpoint and Ralston's method is illustrated in Fig. 2 and Fig. 3, respectively, showing the proportions of susceptible, infected, chronic, and AIDS individuals over time for $h = 0.5$ and $t = 50$ (years).

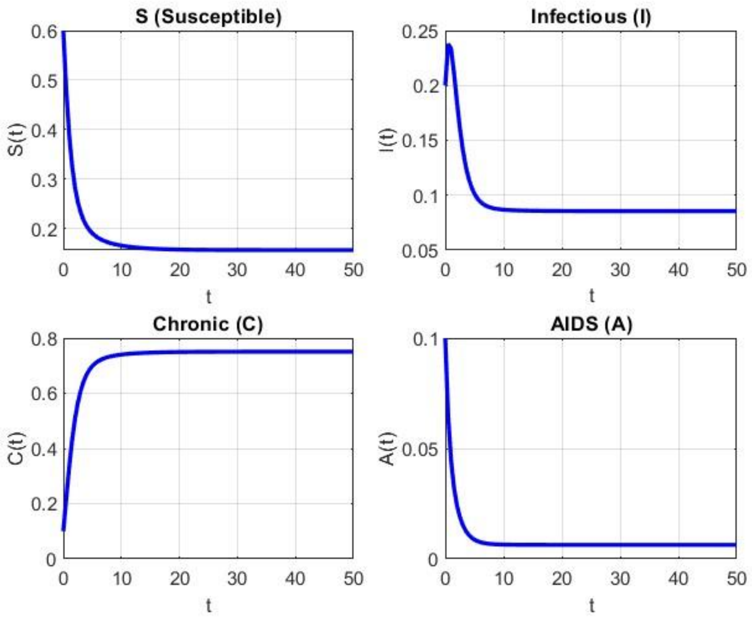

Fig. 2 Graph of the susceptible, infectious, chronic and AIDS subpopulations over time obtained using the Midpoint method

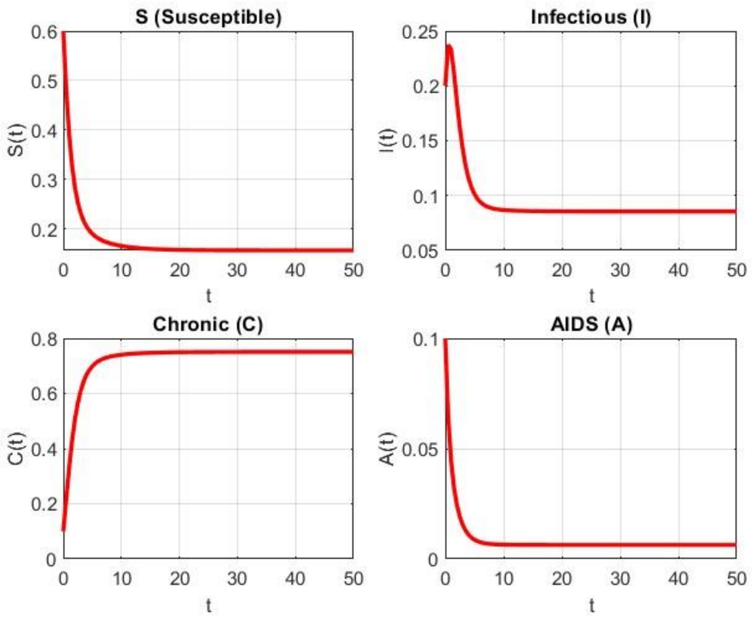

Fig. 3 Graph of the susceptible, infectious, chronic and AIDS subpopulations over time obtained using the Ralston's method

From the graphs, it can be concluded that for both methods, the graphs show nearly identical results. Over time, the susceptible subpopulation declines gradually as more individuals become infected then approaches a stable state. The infected subpopulation initially rises as new infections occurs, then peaks and declines as the number of susceptible individuals diminishes and more individuals transition to the chronic and AIDS stages then stabilizes at a lower level.

The chronic subpopulation rises as individuals progress from infected to chronic but stabilizes as the system reaches an equilibrium. The chronic individuals balanced by the rate at which people transition to AIDS and the rate at which people die. The AIDS subpopulation declines steadily due to the inclusion of factors such as natural death, disease-related mortality, and potentially reduced transmission rates. Similar to the chronic subpopulation, the AIDS subpopulation eventually reaches a steady state as the transition rates stabilize.

3.2 Comparison of the midpoint and Ralston’s methods with the RK4 method.

The performances of the midpoint, Ralston’s and the RK4 methods are depicted graphically in Fig. 4 which showed the proportion of the susceptible, infected, chronic and AIDS subpopulations, respectively, over time for $h = 0.5$ and $t = 50$ (years).

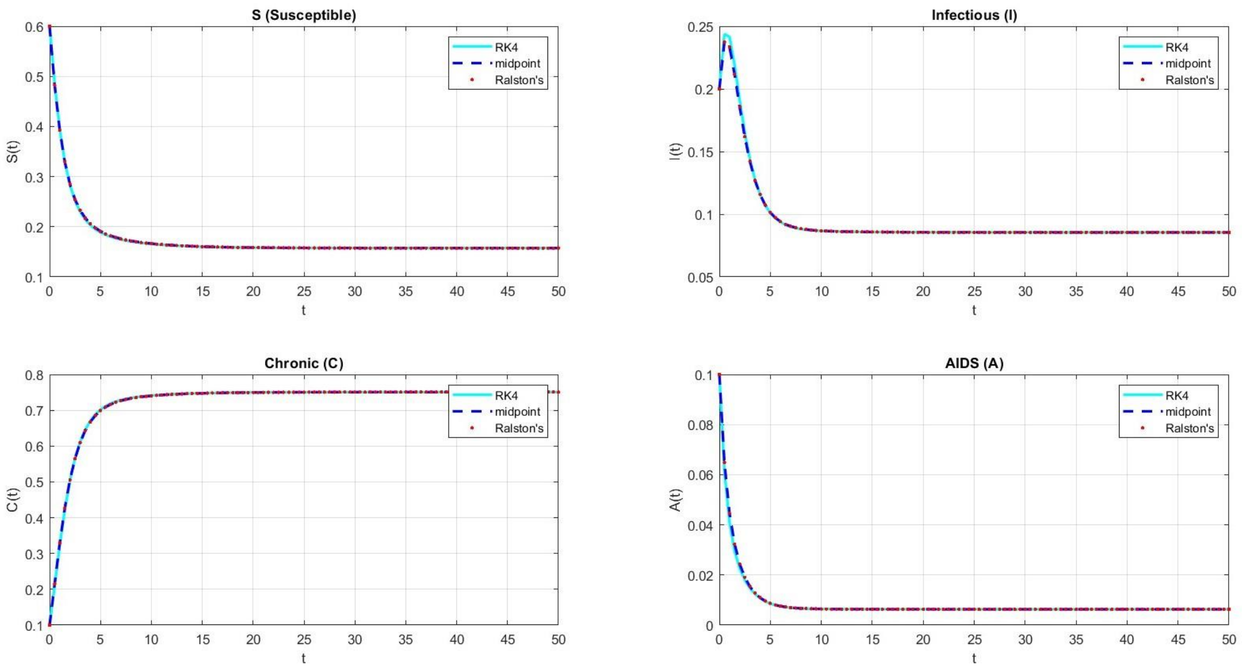

Fig. 4 Graph of the susceptible, infected, chronic and AIDS subpopulation over time obtained using the RK4, midpoint and Ralston’s method.

Table 2 The 2-norm of the difference vector between the results of the RK4 method and the midpoint method, and between the RK4 method and Ralston’s method.

System Variables	$S(t)$	$I(t)$	$C(t)$	$A(t)$
$\|RK4 - Midpoint\|_2$	0.0065102	0.010036	0.0092976	0.0043228
$\|RK4 - Ralston's\|_2$	0.005493	0.009767	0.0083712	0.0045202

Based on the comparison, during *t*=0 to *t*=5, the midpoint and Ralston’s methods did not capture the gradual decrease in the susceptible subpopulation graph as smoothly as the RK4 method. Within the same interval in the infected subpopulation graph, both methods exhibited a slight overshoot during the sharp transition phase. However, in the chronic subpopulation graph, due to the relatively smoother dynamics, both the midpoint and Ralston’s methods performed comparably to the RK4 method. Similar to the susceptible subpopulation graph, both methods failed to capture the decrease as smoothly as the RK4 method in the AIDS population graph.

Based on the graphical results, the midpoint and Ralston’s methods produced outputs that appear nearly identical and almost indistinguishable from those of the RK4 method. However, the analysis of norm metrics revealed that the 2-Norm of the difference vector between the RK4 method and the midpoint method is larger than that between the RK4 method and Ralston's method. This indicates that the difference between the RK4 and midpoint methods is greater than the difference between the RK4 and Ralston's methods. This demonstrates that Ralston's method outperforms the midpoint method. The performance difference arises because the midpoint method, being simpler in implementation, is less optimized. In contrast, Ralston's method employs optimized weights to minimize the global truncation error, resulting in higher accuracy.

4. Conclusion

The results indicate that over time, the proportion of the susceptible subpopulation decreases, the HIV infected subpopulation initially peaks and then declines, the chronic subpopulation increases, and the AIDS subpopulation decreases. All subpopulations eventually stabilize as the system reaches equilibrium. The graphs obtained using midpoint and Ralston's methods were compared with the existing solution obtained from the RK4 method. The comparison reveals that the results from both the midpoint and Ralston's methods are nearly identical and almost indistinguishable from those obtained using the RK4 method, except in regions with rapid transitions.

In these regions, short-term discrepancies were observed between midpoint and Ralston's methods with the RK4 method. Both the midpoint and Ralston's methods were less accurate in capturing the rapid transitions compared to the RK4 method. An analysis of the norm metrics reveals that Ralston's method outperforms the midpoint method. The performance difference arises because the simpler, less optimized midpoint method contrasts with Ralston's method, which uses optimized weights to reduce error and improve accuracy. These findings suggest that for computational efficiency, Ralston's method provides a reasonable compromise, while the midpoint method is better suited for simple scenarios or preliminary testing. Nevertheless, both the midpoint and Ralston's methods have proven to be effective approaches for solving the SICA model.

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Conflict of Interest

Authors declare that there is no conflict of interest regarding the publication of the paper.

Author Contribution

The authors confirm contribution to the paper as follows: ***study conception and design:*** *Nur Syahirah Saari;* ***solve the equations:*** *Nur Syahirah Saari;* ***analysis and interpretation of results:*** *Nur Syahirah Saari, Azila Md. Sudin;* ***draft manuscript preparation:*** *Nur Syahirah Saari, Azila Md. Sudin. All authors reviewed the results and approved the final version of the manuscript.*

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