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# Homotopy Analysis Method to Solve Second-Order Nonlinear Ordinary Differential Equations

**Nur Nasrin Athirah Idris**[1], **Mahathir Mohamad**[1,*],**Fazlina Aman**[1]

[1]Department of Mathematics and Statistics, Faculty of Applied Sciences and Technology,
Universiti Tun Hussein Onn Malaysia, Johor, MALAYSIA

* Corresponding Author Designation

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**Abstract**: In this research, a homotopy analysis method (HAM) is used for solving second-order nonlinear ordinary differential equations (ODEs). The approximate and complex solutions of the second-order nonlinear ODEs problem were solved using Maple software. The solution easier to solve and the computational works will be reduced when using Maple software. The numerical solution that has obtained using HAM is being compared with the exact solution and also being compared with the adomian decomposition method (ADM) to determine the efficiency and accuracy of the HAM towards the exact solution. The convergence of the HAM and the absolute error is discussed further in this research. The results for the homotopy analysis method were obtained using Maple 2015. It was observed that the homotopy analysis method and the adomian decomposition method were efficient in solving second-order nonlinear ODE. However, a modified homotopy analysis method (MHAM) can be used to obtain an approximate solution close to the exact solution.

**Keywords**: Homotopy Analysis Method, Adomian Decomposition Method, Ordinary Differential Equations

## 1. Introduction

A nonlinear differential equation has been explored by mathematicians and researchers with approaches, and tools. Nonlinear ODEs appears in the study of number of branches of applied mathematics such as rheology, quantitative biology, physiology, electrochemistry, scattering theory, diffusion transport theory, potential theory, and elasticity [1]. In the previous researches, mathematicians, and researchers introduced many methods to obtain an approximate solution for the second-order nonlinear ODEs. Some of the methods that used by the researcher to get the solutions

**Corresponding author: mahathir@uthm.edu.my*

are the homotopy perturbation method (HPM), homotopy asymptotic method, variational iteration method (VIM), and runge kutta method (RKM).

In this study, the second order nonlinear equation is considered as

$$y'' = f(x, y, y'), \quad \text{Eq. 1}$$

with the initial condition

$$y(0) = y' = 0 \quad \text{Eq. 2}$$

[4] introduced his idea where he proposed a homotopy analysis method (HAM) using the concept of fundamental in topology and differential geometry. He applied it to solve a lot of nonlinear in science, finance and engineering problems. HAM is an independent variable between any of small or large physical parameters at all [5].

The aim of this paper is to apply HAM to obtain the approximate solutions of second-order nonlinear ordinary differential equations. We demonstrate the accuracy and efficiency of the HAM through some test examples. Numerical comparison will be made against the adomian decomposition method (ADM).

## 2. Methodology

In order to describe HAM, we consider the following differential equation

$$N[u(x)] = 0, \quad \text{Eq. 3}$$

where $N$ is a nonlinear operator $u$ and $x$ denote the independent variable and $u(x)$ is an unknown function. By means of generalizing the traditional homotopy method, we construct the zero-order deformation equation,

$$(1 - q)L[\varphi(x; q) - u_0(x)] = q\hbar\{N[\varphi(x; q)]\}, \quad \text{Eq. 4}$$

where $q \in [0,1]$ is the embedding parameter,

$\hbar$ is a nonzero auxiliary function,

$L$ is an auxiliary linear operator,

$u_0(x)$ is an initial guess of $u(x)$,

and $\varphi(x; q)$ is an unknown function,

It is important to note that in HAM, it has great freedom to choose auxiliary objects such as $\hbar$and $L$. Obviously, when the embedding parameter $q = 0$ and $q = 1$, both

$$\varphi(x; 0) = u_0(x), \varphi(x; 1) = u(x), \quad \text{Eq. 5}$$

hold. Thus as $q$ increase from 0 to 1, the solution $\varphi(x; q)$ varies from the initial guess $u_0(x)$ to the solutions $u(x)$. Expanding $\varphi(x; q)$ in Taylor series with respect to $q$, one has

$$\varphi(x; q) = u_0(x) + \sum_{m=1}^{+\infty} u_m(x)q^m, \quad \text{Eq. 6}$$

where

$$u_m = \frac{1}{m!}\frac{\partial^m \varphi(x;q)}{\partial q^m}|q = 0. \quad \text{Eq. 7}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are so properly chosen, then the series Eq. 6 converges at $q = 1$ one has

$$\varphi(x; 1) = u_0(x) + \sum_{m=1}^{+\infty} u_m(x), \quad \text{Eq. 8}$$

which must be one of the solutions of the original nonlinear equation. If $\hbar = -1$ (Eq. 4) becomes

$$(1-q)L[\varphi(x;q)-u_0(x)]+q\{N[\varphi(x;q)]\}=0. \qquad \text{Eq. 9}$$

According to Eq. 7, the governing equations can be deduced from the zero order deformation Eq. 4. We define the vectors as

$$\vec{u}=\{u_0(x),u_1(x),\dots,u_i(x)\}. \qquad \text{Eq. 10}$$

Differentiating Eq. 4 $m$ times with respect to embedding parameter $q$ and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called *mth*-order deformation equation:

$$L[u_m(x)-\chi_m u_{m-1}(x)]=\hbar R_m(\overrightarrow{u_{m-1}}), \qquad \text{Eq. 11}$$

where

$$R_m(u_{m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}\{N[\varphi(x;q)]\}}{\partial q^{m-1}}|q=0, \qquad \text{Eq. 12}$$

and $$\chi_m=\begin{cases}0, m\le 1,\\ 1, m>1.\end{cases}$$

It should be emphasized that $u_m(x)$ $m\ge 1$ are governed by the linear Eq. 11 with the linear boundary conditions that come from the original problem, which can be easily solved by symbolic computation software such as Matlab or Maple. The uniqueness of HAM is depending on the solution obtain from Eq. 3 and if it produce a unique solution, then it can be prove that the HAM is more accuracy and effectiveness. If it equation 3 does not produce unique solution, the HAM will give a solution from other possible solution.

## 3. Results and Discussion

For this research, the homotopy analysis method will be focus on solving second-order nonlinear ordinary differential equations (ODEs). The problems that will be solved are taking from the articles by [1] and [2] to obtain the solutions. Next, the solutions obtained from the homotopy analysis method (HAM) will be compared with the adomian decomposition method (ADM).

3.1 Example 1

Consider the second-order nonlinear ordinary differential equation

$$y''(x)-y(x)=20\le x\le 10, \qquad \text{Eq. 13}$$

subject to initial conditions

$$y(0)=y'(0)=0, \qquad \text{Eq. 14}$$

which has the exact solution

$$y(x)=e^{x}+e^{-x}-2, \qquad \text{Eq. 15}$$

Furthermore, Eq. 13 suggests that we define the linear operator as [4]

$$L[\varphi(x;q)]=\frac{\partial^2\varphi(x;q)}{\partial x^2}, \qquad \text{Eq. 16}$$

and nonlinear operator as

$$N[\varphi(x;q)] = \frac{\partial^2 \varphi(x;q)}{\partial x^2} - \varphi(x;q) - 2, \qquad \text{Eq. 17}$$

Next, substitute equations Eq. 16 and 17 into zero order deformation equation as in Eq. 4.
So we get,

$$(1-q)\left[\varphi''(x;q) - y_0(x)\right] = q\hbar\left[\varphi''(x;q) - \varphi(x;q) - 2\right], \qquad \text{Eq. 18}$$

Then by using definition at Eq. 8, we get

$$\begin{aligned}&(1-q)\left[y_1''(x)q + y_2''(x)q^2 + y_3''(x)q^3 + \cdots\right] = \\ &qh\begin{bmatrix}\left[y_0''(x) + y_1''(x)q + y_2''(x)q^2 + y_3''(x)q^3 + \cdots\right] - \\ \left[y_0(x) + y_1(x)q + y_2(x)q^2 + y_3(x)q^3 + \cdots\right] - 2\end{bmatrix}\end{aligned} \qquad \text{Eq. 19}$$

By using Maple software, differentiate Eq. 19 with respect to $q$ to get the first derivative and then put $q = 0$, we obtain

$$y_1''(x) = \hbar\left[-y_0(x) + y_0''(x) - 2\right], \qquad \text{Eq. 20}$$

So we get,

$$y_1(x) = -\hbar x^2, \qquad \text{Eq. 21}$$

To get $y_2(x)$, we differentiate the first derivative with respect to $q$ and then put $q = 0$ by using Maple software.

Now, we successively obtain

$$\begin{aligned}&y_1(x) = -\hbar x^2,\\ &y_2(x) = \frac{\hbar^2 x^4}{12} - \hbar^2 x^2 - \hbar x^2,\\ &\boldsymbol{y_3(x) = \frac{\hbar^3 x^6}{360} + \frac{\hbar^3 x^4}{6} + \frac{\hbar^2 x^4}{6} - \hbar^3 x^2 - 2\hbar^4 x^2 - \hbar x^2},\\ &\quad\vdots\end{aligned} \qquad \text{Eq. 22}$$

Then the series solution expression can be written in the form of

$$\boldsymbol{y(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \cdots} \qquad \text{Eq. 23}$$

and so forth. Hence, the series solution when $\hbar = -1$ is

$$y(x) \approx x^2 + \frac{x^4}{12} + \frac{x^6}{360} + \frac{x^8}{20160} + \cdots \qquad \text{Eq. 24}$$

Then, we get the closed form which is

$$y(x) = e^x + e^{-x} - 2, \qquad \text{Eq. 25}$$

which is same with the exact solution in Eq. 15.

**Table 3.1 Numerical solution of homotopy analysis method (HAM) and adomian decomposition method (ADM) compared with the exact solution**

| x | Exact Solution | Solution with HAM | Solution with ADM [1] |
|---|---|---|---|
| 0.1 | 0.010008336 | 0.010008336 | 0.010008336 |
| 0.2 | 0.040133511 | 0.040133511 | 0.040133512 |
| 0.3 | 0.090677028 | 0.090677029 | 0.090677028 |
| 0.4 | 0.162144744 | 0.162144744 | 0.162144743 |
| 0.5 | 0.255251930 | 0.255251931 | 0.255251930 |
| 0.6 | 0.370930436 | 0.370930436 | 0.370930436 |
| 0.7 | 0.510228011 | 0.510338011 | 0.510228011 |
| 0.8 | 0.674869893 | 0.674869892 | 0.674869892 |
| 0.9 | 0.886172771 | 0.866172771 | 0.886172770 |
| 1.0 | 1.086161270 | 1.086161269 | 1.086161269 |

**Table 3.2 Numerical solution of adomian decomposition method (ADM) compared with the exact solution**

Exact solution = $y(x) = e^{x} + e^{-x} - 2$

| x | Solution with HAM | Solution with ADM [1] |
|---|---|---|
| 0.1 | 0.000000000 | 0.000000000 |
| 0.2 | 0.000000000 | 0.000000001 |
| 0.3 | 0.000000001 | 0.000000000 |
| 0.4 | 0.000000000 | 0.000000001 |
| 0.5 | 0.000000001 | 0.000000000 |
| 0.6 | 0.000000000 | 0.000000000 |
| 0.7 | 0.000000000 | 0.000000000 |
| 0.8 | 0.000000001 | 0.000000001 |
| 0.9 | 0.000000000 | 0.000000001 |
| 1.0 | 0.000000001 | 0.000000001 |

3.2 Example 2

Consider the second-order nonlinear ordinary differential equation

$$y''(x) - y'(x) = 0, \qquad \text{Eq. 26}$$

subject to initial conditions

$$y(0) = y'(0) = 1, \qquad \text{Eq. 27}$$

which has the exact solution

$$y(x) = e^{x}. \qquad \text{Eq. 28}$$

Furthermore, Eq. 26 suggests that we define the linear operator as [4]

$$L[\varphi(x;q)] = \frac{\partial^2 \varphi(x;q)}{\partial x^2}, \qquad \text{Eq. 29}$$

and nonlinear operator as

$$N[\varphi(x;q)] = \frac{\partial^2 \varphi(x;q)}{\partial x^2} - \frac{\partial \varphi(x;q)}{\partial x}, \qquad \text{Eq. 30}$$

Next, substitute Eq. 29 and 30 into zero order deformation equation as in Eq. 4
So we get

$$(1-q)\left[\varphi''(x;q) - y_0(x)\right] = q\hbar\left[\varphi''(x;q) - \varphi'(x;q)\right], \qquad \text{Eq. 31}$$

Then by using definition at Eq. 8, we get

$$\begin{gathered}(1-q)\left[y_1''(x)q + y_2''(x)q^2 + y_3''(x)q^3 + \ldots\right] = \\ q\hbar\begin{bmatrix}\left[y_0''(x) + y_1''(x)q + y_2''(x)q^2 + y_3''(x)q^3 + \ldots\right] - \\ \left[y_0'(x) + y_1'(x)q + y_2'(x)q^2 + y_3'(x)q^3 + \ldots\right]\end{bmatrix}.\end{gathered} \qquad \text{Eq. 32}$$

By using Maple software, differentiate Eq. 32 with respect to $q$ to get the first derivative and then put $q = 0$, we obtain

$$y_1''(x) = \hbar[y_0''(x) - y_0'(x), \qquad \text{Eq. 33}$$

So, we get

$$\mathrm{y}_1(\mathrm{x}) = \frac{-\hbar \mathrm{x}^2}{2}, \qquad \text{Eq. 34}$$

To get $y_2(x)$**,** we differentiate the first derivative with respect to q and then put $q = 0$ by using Maple software.

Now, we successively obtain

$$y_1(x) = \frac{-\hbar x^2}{2},$$

$$y_2(x) = \frac{\hbar^2 x^3}{6} - \frac{\hbar^2 x^2}{2} - \frac{\hbar x^2}{2},$$

$$y_3(x) = -\frac{\hbar^3 x^4}{24} + \frac{\hbar^3 x^3}{3} - \frac{\hbar^3 x^2}{2} - \frac{\hbar x^2}{2} + \frac{\hbar^2 x^3}{3} - \hbar^2 x^2. \qquad \text{Eq. 35}$$

$$\vdots$$

Then the series solution expression can be written in the form of

$$y(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \cdots \qquad \text{Eq. 36}$$

and so forth. Hence, the series solution when $\hbar = -1$ is

$$y(x) \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \qquad \text{Eq. 37}$$

Then, we get the closed form which is

$$y(x) = e^{x}. \qquad \text{Eq. 38}$$

which is same with the exact solution in Eq. 28.

**Table 3.3 Numerical solution of homotopy analysis method (HAM) and adomian decomposition method (ADM) compared with the exact solution**

| x | Exact Solution | Solution with HAM | Solution with ADM [2] |
|---|---|---|---|
| 0 | 1.000000000 | 1.000000000 | 1.000000000 |
| 0.1 | 1.105170918 | 1.105170918 | 1.105170918 |
| 0.2 | 1.221402758 | 1.221402758 | 1.221402758 |
| 0.3 | 1.349858808 | 1.349858808 | 1.349858808 |
| 0.4 | 1.491824698 | 1.491824698 | 1.491824698 |
| 0.5 | 1.648721271 | 1.648721271 | 1.648721270 |

**Table 3.4 Numerical solution of adomian decomposition method (ADM) compared with the exact solution**

Exact solution = $y(x) = e^{x}$

| x | Solution with HAM | Solution with ADM [2] |
|---|---|---|
| 0 | 0.000000000 | 0.000000000 |
| 0.1 | 0.000000000 | 0.000000000 |
| 0.2 | 0.000000000 | 0.000000000 |
| 0.3 | 0.000000000 | 0.000000000 |
| 0.4 | 0.000000000 | 0.000000000 |
| 0.5 | 0.000000000 | 0.000000001 |

3.3 Discussion

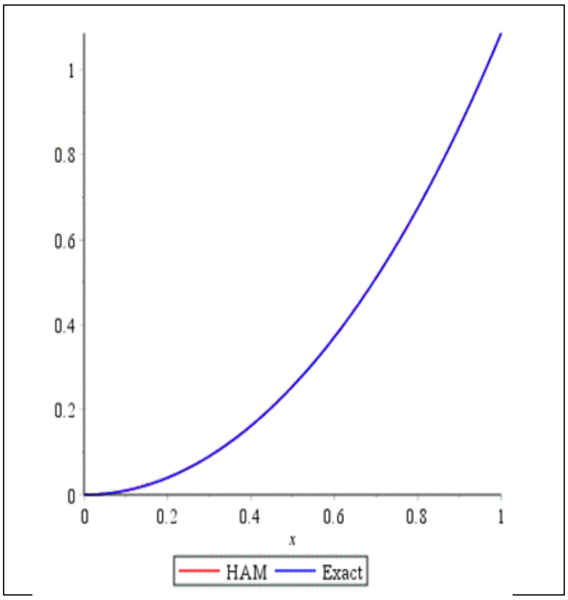


**Figure 3.1 Graph of numerical solutions of homotopy analysis method compared with exact solution for Example 1**

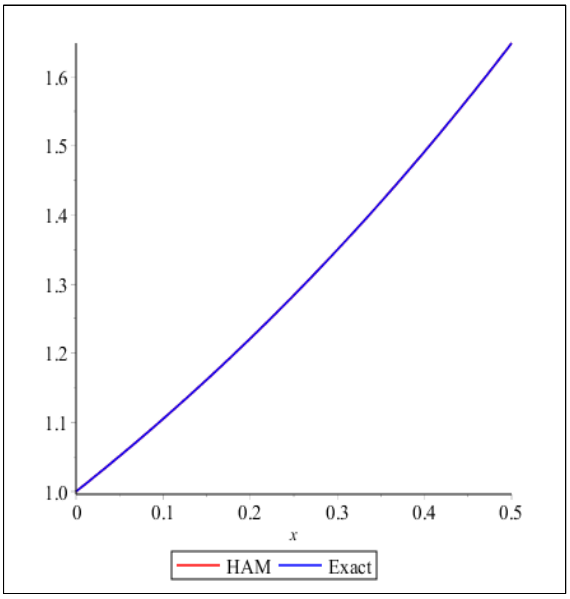


**Figure 3.2 Graph of numerical solutions of homotopy analysis method compared with exact solution for Example 2**

Based on Figure 3.1 and Figure 3.2, it can be concluded that HAM is efficient and accurate to the exact solution because the graph does not show any differences between HAM and exact solution. Based on the example that we solved on the previous section, we successfully obtained an approximate solution. So, it shows that HAM able to solve second-order nonlinear ordinary differential equations. However, if we used the optimal homotopy asymptotic method (OHAM) which is an improve adaptation of HAM the solutions obtain will be more efficient and accurate because OHAM will involve simpler integration and less computation than the standard HAM.

## 4. Conclusion

Firstly, the background of HAM and second-order nonlinear ODEs is being explored for the research. Then, a standard method of HAM to solve second-order nonlinear ODEs also have been studied. Two examples of second-order nonlinear ODEs are solved using the homotopy analysis method (HAM) and the solutions obtained being compared with the adomian decomposition method (ADM) in finding the accuracy of the methods with the exact solution. Maple R2015 was the software used which provide numerical solutions and also graphical output that make the research easier to solve. The numerical solutions obtained from HAM are making small difference from the exact solutions and for ADM, the solutions obtained also make a small difference with the exact solutions. The graphical output shows the relation of exact solutions and HAM. From these two examples, we can say that both HAM and ADM make a small difference in the absolute error. So, it can be concluded that both HAM and ADM can be used to solve second-order nonlinear ordinary differential equations (ODEs) because both methods are accurate with the exact solution. We can also conclude that HAM provides a convenient way of controlling the convergence of approximation series, which is a fundamental qualitative difference in analysis between HAM and other methods.

There are some recommendations to improve the homotopy analysis method to get more accurate solution;

1. Using modified homotopy analysis method (MHAM) to get the solutions close with the exact solution
2. The parameter $\hbar$, can be improved which make it optimally recognize and will converge faster

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**References**

[1] Agom, E. U., & Badmus, A. M. (2015). Application of Adomian Decomposition Method in Solving Second Order Nonlinear Ordinary Differential Equations. *International Journal of Engineering Science Invention*, *4*(11), 60–65.

[2] Sadeghinia, A. (2015). *One Solution for the System of Second Order ODEs by ADM and Application of Physics Problems*. *2*(2), 1–10.

[3] Liao, S. (2011). Homotopy analysis method in nonlinear differential equations. Homotopy Analysis Method in Nonlinear Differential Equations, 9783642251320, 1–565. https://doi.org/10.1007/978-3-642-25132-0.

[4] Turq, S. (2020). Homotopy type methods for nonlinear differential equations Saed Turq. June. https://doi.org/10.13140/RG.2.2.23905.12644.

[5] Ahmed, M., Rafiq, A., Ahmed, M., & Hussain, S. (2008). A general approach to specific second order ordinary differential equations using homotopy perturbation method A general approach to specific second order ordinary differential equations using homotopy perturbation method. February 2019. https://doi.org/10.1016/j.physleta.2008.05.070.