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The Study of Population Dynamics by Delay Differential Equations

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Abstract: Population dynamics model is one of the important topics in mathematical modeling. The purpose of using these models is to generate a model that can well describe the population species at any moment. Nowadays, this technique has been well developed and not only effective in the form of an ordinary differential equation but also derived a new branch, a delayed population dynamics model which is a delay differential equation. In the delay differential equation, the time delay is the main core for the equation and this factor is always represented as the time lag taken between the implementation of control and responding of the system. In this study, two population dynamics models were investigated correspond to the form of ordinary and delay differential equations. The comparison between these models were conducted and this results in no significant changes if the value of time delay is small enough, but the solution will meet a great change on it which is the rise of oscillation if the time delay keeps increasing. There is a specific name for the phenomenon, called Hopf-bifurcation. Therefore, the determination to obtain the critical value of time delay was taken in order to know when the Hopf-bifurcation will occur. According to the result, once the parameter achieves the critical value, the initially stable equilibrium has become unstable which was a loss of stability and then lead to the happening of Hopf-bifurcation which was in the form of periodic solution.

Keywords: Population Dynamics, Delay Differential Equation, Ordinary Differential Equation, Time Delay, Hopf-Bifurcation

1. Introduction

Population dynamic is one of the applications of mathematical modeling that made an important role in real life especially in biology and ecology. The related models are often used in order to predict the trend of population, the number of infections or others that are related to the population investigated and these models are expected to provide an associated prediction and description to the present environment or conditions [1]. The first population dynamic model which is called the exponential growth model was introduced, and it is a simple linear ordinary differential equation (ODE) [2].

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$$\frac{dP}{dt} = aP; \text{ where } P(t) = P_o e^{at}. \qquad Eq.1$$

From this model, the population at any moment t mainly depends on the intrinsic growth rate a, which is the only factor that will affect the trend and behavior of the solution. There is a fact which pointed out and indicates that there will be no limitation for the population growth up due to the exponential term. Therefore, a modification for the first population model was taken and a new factor was introduced based on the environmental consideration so that the resources available in the corresponding environment are involved for the model generation. As a result, the logistic growth model was introduced, where the new parameter K is represented as the environment carrying capacity [3].

$$\frac{dP}{dt} = a\left(1 - \frac{P(t)}{K}\right)P(t). \qquad Eq.2$$

The environment carrying capacity specified that there is a limited value of resources that can be supplied for the corresponding population. The carrying capacity of an environment depends on the factors like adequate food, shelter, water, and mates. Over time, the population size will be unchanged at the end and converge at a state level which is equal to the value of carrying capacity. It is reasonable to say that the population dynamic can be better represented by the logistic growth model after the modification. The model can better describe the population if there is an ideal condition with no time delays occur but it is impossible.

Time delay is a real-life factor that should be considered in the application of ODE and the presence of the time delays may cause the difference between the model description and the real-world phenomenon. Therefore, the knowledge of delay differential equation (DDE) is necessary and essential for the dynamical system, it also can be applied to other technological control problems and dynamical models. In definition, the derivative for DDE at any time depends on the solution at the previous time. Yang [4] depicted the simplest constant delay equation has the form of

$$y'(t) = f\big(t, y(t), y(t-\tau)\big), \text{ where} \qquad Eq.3$$

the differential equation that depends on the value of y in the past with a certain state and the parameter τ is the time delay. In ecology, the time delay is often used to represent the resource regeneration time, maturation period, or feeding time, all the processes that take a certain time and cannot complete instantaneously. For example, Vaidya and Wu [5] used the time delay to represent the riskless time of budworms to the environment which is about 9 months that during the state from egg to second instar caterpillar, and the model is used for better outbreak control. Moreover, a modification for logistic growth model based on properties of DDE was introduced [6], and the model is more realistic and reliable to describe natural conditions such that $Eq.2$ becomes

$$\frac{dP}{dt} = a\left(1 - \frac{P(t-\tau)}{K}\right)P(t), \qquad Eq.4$$

The population growth description can be better represented by a time-delayed model compared to the ODE because organisms rarely react instantaneously to the changes in the system. For example, Freitas [7] had shown that the trajectory path exponential growth model with delay is grow more slowly compared with the model without delay and the accuracy is enhanced. In ODE, a fact about the population condition for the species in the past may be ignored, the growth rate or rate of change of the population at time t only depends on the relative number of individuals at that current time. The critical factor that was not taken into account for these population models is the existence of factor time delay in real life. Some processes in ecology may not be presented or completed in the form of instantaneous, such as human gestation and reproduction. Therefore, the knowledge of the delay differential equation is essential for the modification of the population dynamic model to make the model more appreciated the real-life conditions.

For the models which have incorporated the time delay factor, the current system behavior may vary with the value of delay. In most cases, the system stability for the model is changed from stable to unstable when the value of the delay gets larger, which then induces the happening of a bifurcation. Bifurcation is a phenomenon where the solution changes drastically due to the parameter that at a certain value. In addition, there is a variation in the population description between the population models with and without delay in terms of population growth trends. Therefore, this study is proposed to analyse the system behavior of a delayed population model and make a comparison with the model without delay.

2. Preliminaries

2.1 Delay Differential Equation

A differential equation that the derivative of a function is depending on the values of the function in the previous times but different with the ODE ones in terms of the value of the function at current times. The functions in the previous time is referred as the delayed function and a new parameter τ which represents the time delay was added. Besides, the initial condition of DDE must be specified for the time interval $[0\ ,\tau]$ that is based on time delay used.

2.1 Equilibrium

An equilibrium is the level where the state variable does not change once the solution attains the level or it converges at the level in the end. Therefore, the rate of change is considered as equal zero at that level. Equilibrium can be classified as stable or unstable for the situation of system is moving toward or away to the equilibrium as time over. Generally, the stable equilibriums are called 'attractor' and the unstable equilibriums are called 'repeller' for the dynamical system.

2.2 Stability Analysis

Stability analysis is used to determine the stability of the equilibrium and to obtain the trajectory path or system behavior over time. In application of DDE, the analysis is mainly used to determine the region in the delay parameter space at which the system is still stable.

2.3 Dimensionless Analysis

Dimensionless analysis is used to convert a dimensional model to dimensionless form does not get any affected on its original meaning. A model equation that expressed in dimensionless term by reducing of the number of parameters in the equation is helpful in the procedure of stability analysis and it can better illuminate the relationships between parameters.

2.4 Hopf-bifurcation

Hopf-bifurcation is an extension of the bifurcation theory and the corresponding techniques is modified for the application of delay differential equation [8]. [9] had stated that the stability of the models with DDE is found that easily affected by the time delay parameter and it is common to see that a stable equilibrium of delay differential equation becomes unstable due to the value of time delay becomes larger. This condition provides the dynamics of DDE are extremely sensitive to the parameter and more complex dynamics will be generated when increasing time delay. In other words, Hopf-bifurcation occurs when the periodic solution or limit cycle surrounding an equilibrium is risen or disappeared as the time delay varies. Therefore, the Hopf-bifurcation theorem is one of the crucial parts for DDE which is used to establish the existence of period solutions.

3. Results and Discussion

3.1 Exponential model with and without delay

The exponential growth or decay model introduced by Malthus [2] is written as

$$u'(t) = \alpha u(t), \qquad Eq.5$$

and the corresponding model with delay is

$$u'(t) = \alpha u(t-\tau), \qquad Eq.6$$

where α is the intrinsic rate that can be positive or negative, since it is depending on whether it is used for exponential growth model or exponential decay model respectively. In order to see the difference between ODE and DDE, the graphs in Figure 1 were generalized by these equations. It was assumed that the initial value for $Eq.5$ is $u(0) = 100$. For the case of DDE, it may a bit different compared with ODE, the initial condition was specified for time interval $[0\,,\tau]$ but not for time of 0 only, to make the comparison significantly, $u = 100$ was used for $[0\,,\tau]$, the parameter α was set with $\alpha = -1$ which means that there are decaying for both equations.

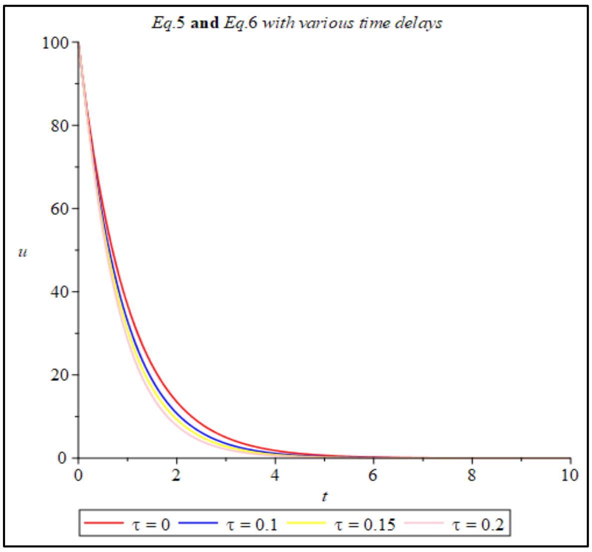

Figure 1: $Eq.5$ and $Eq.6$ with various time delays

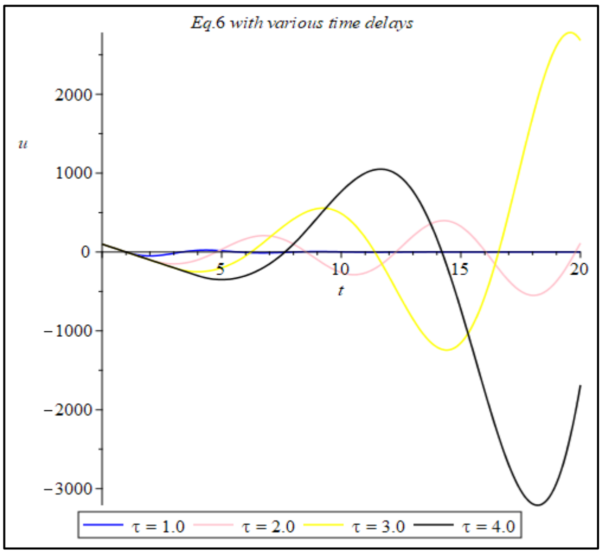

Figure 2: $Eq.6$ with various time delays

Based on the first graph, if the value of time delay is sufficiently small like 0.1, 0.15, and 0.2, then all the solutions with delay are looks very similar with to solution without delay. This implies that the smaller time delay is not giving the solution result in a large difference and only result in growth or decay more slowly compared with the condition without time delay. However, if the time delay kept increasing, it might be leading to a huge change for the solution of $Eq.6$ and induced the arise of oscillation over time.

From Figure 2, it can be seen that there are huge changes for the solution over time as the time delay increases. When $\tau = 1$ or less, the solution (blue color) has finally decayed and converged to 0, and there was a stable equilibrium or stable steady state be set as the solution is moving toward it. However, when $\tau = 2$ or more, the solutions (another three curves in Figure 2) had started to decay with oscillation and the periodic solutions had occurred which means that there was a loss of stability for the stable equilibrium. The situation indicates that the loss of stability and the occurrence of periodic motion is called Hopf-bifurcation. As the bifurcation occurs, the stable equilibrium becomes unstable, and there is a critical value of the time delay τ, which leads to the situation happening. Once the time delay is equal to or greater than the critical value, the Hopf-bifurcation happens.

Next, it was to proceed with the stability analysis in order to know when the equilibrium will be in a steady-state and when will not. However, it should take the dimensionless analysis first for the delayed model but not the ordinary model, this is because it is an infinitely dimensional equation with high complexity and the $Eq.5$ is already in simplest form. Therefore, two equations were introduced for scaling of $Eq.6$ which are $U(\omega) = u(t)$ and $\omega = nt$, where $n > 0$ and the $Eq.6$ will become

$$\begin{aligned} \frac{du}{d\left(\frac{\omega}{n}\right)} &= \alpha u\left(\frac{\omega}{n} - \tau\right) \\ U'(\omega) &= \alpha\left(\frac{1}{n}\right)U(\omega - \tau n) \end{aligned}$$

Two new parameters were introduced to reduce the number of parameters: $\tau = \frac{1}{\mathrm{n}}$ and $\beta = \alpha\tau$. After that, the dimensionless analysis is completed.

$$U'(\omega) = \beta U(\omega - 1) \qquad Eq.7$$

Equilibrium will be determined as well as this will be used for the stability test. As the equilibrium state is achieve, there will be no changes anymore, the rate of change will be 0. This implies that function $U(\omega - 1)$ will no difference with the value of $U(\omega)$ in equilibrium. The equilibrium for both equations can be simply obtained by just making the rate of change equal to 0. As the result, a trivial equilibrium was obtained in both model $Eq.5$ and $Eq.6$.

3.2 Stability Test: Exponential model with delay and without delay

For the exponential decay model without delay, the stability of trivial equilibrium is known to depend on the value of α, if negative then there is a stable equilibrium, if positive the equilibrium is stable. No bifurcation happened because of the existence of the parameter. However, as the Figure 2 shows, the stability of the equilibrium was changed due to the value of time delay was getting larger.

To determine the stability, it was assumed that there was an exponential solution which is $U(\omega) = Ce^{\lambda\omega}$ for $Eq.7$, and substituted it inside the equation. As the sequence, a corresponding characteristic equation was obtained.

$$\begin{aligned} C\lambda e^{\lambda\omega} &= \beta C e^{\lambda(\omega-1)} \\ \lambda - \beta e^{-\lambda} &= 0 \qquad Eq.8 \end{aligned}$$

There are two possibilities for the eigenvalue of the characteristic equation which are real or imaginary. For the imaginary eigenvalues, considered that the eigenvalue is in the form of $\lambda = x + iy$, where x was a real part and y was the imaginary part. Substitute the complex eigenvalue proposed into the characteristic equation, the $Eq.8$.

$$\begin{aligned} (x + iy) - \beta[e^{-(x+iy)}] &= 0 \\ x + iy &= \beta e^{-(x+iy)} \\ &= \beta e^{-x}[\cos(-y) + isin(-y)] \\ x + iy &= \beta e^{-x}\cos(y) - i\beta e^{-x}sin(y) \qquad Eq.9 \end{aligned}$$

From the $Eq.9$, the general equations for the real and imaginary parts was found:

$$\begin{aligned} x &= \beta e^{-x}\cos(y) \qquad Eq.10 \\ y &= -\beta e^{-x}sin(y) \qquad Eq.11 \end{aligned}$$

Recall that the equilibrium is stable if and only if all the eigenvalues have negative real parts but unstable if there are any positive real parts. Besides, from the $Eq.10$ and 11, it is known that the real and imaginary parts of eigenvalue is depending on the value of β. That is, what value of β is which there is x equal to zero, and there are existences of a pure imaginary eigenvalue. Let assumed that $x = 0$ and $y \neq 0$ for $Eq.10$,

$$\begin{aligned} 0 &= \beta e^{-(0)}\cos(y) \\ \mathrm{y} &= \pm\frac{\pi}{2} + k\pi, k = 0, 1, 2, \ldots \qquad Eq.12 \end{aligned}$$

And then, use the information obtained to determine the value of β,

$$\pm\frac{\pi}{2} = -\beta e^{-(0)} sin\left(\pm\frac{\pi}{2}\right)$$
$$\beta = \pm\frac{\pi}{2} \qquad Eq.13$$

The negative β was chosen as the critical value for the happening of Hopf-bifurcation. This is because the equilibrium is initially in steady state which refer to Figure 1 and 2, means that there were all negative eigenvalues. If the positive one was chosen, $\beta = \frac{\pi}{2}$, the $Eq.10$ would become positive, the real eigenvalue were positive. Once the value of β arrives at the critical value, $-\frac{\pi}{2}$, the purely complex eigenvalues (no real eigenvalues) are formed and the solution will become unstable starting from this point due to the arise of periodic solution.

Based on the finding above, three conditions were concluded for $Eq.8$ given that there were complex conjugate eigenvalues. Recall that parameter β actuary was the product of α and τ.

1. If $-\frac{\pi}{2} < \alpha\tau < 0$, then the equilibrium $U = 0$ is stable due to the real part is smaller that 0 for all eigenvalues.
2. If $\alpha\tau = -\frac{\pi}{2}$, then the real part is equal to 0 and there are exist of purely complex conjugate eigenvalues, $\lambda = \pm i\frac{\pi}{2}$. Hopf bifurcation is occurs.
3. If $\alpha\tau < -\frac{\pi}{2}$, there are eigenvalues with real parts greater than 0 and the equilibrium $U = 0$ is unstable.

Next, the real eigenvalue for the characteristic equation was analyzed by using a graph which consist of 2 equation that are $z = \lambda$ and $z = \beta e^{-\lambda}$, in order to determine where the point of intersection located.

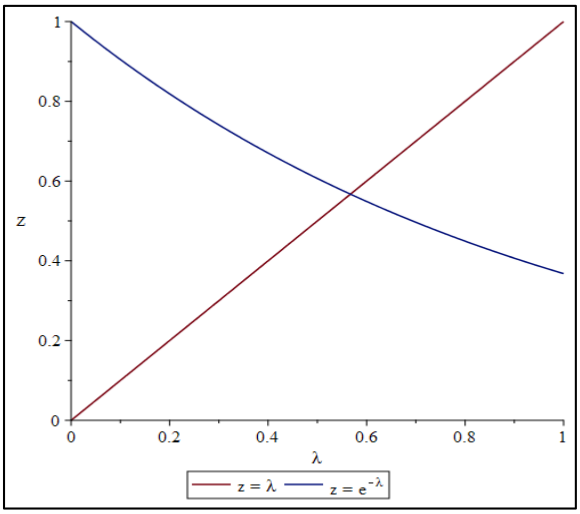

Figure 3: For real and positive eigenvalues

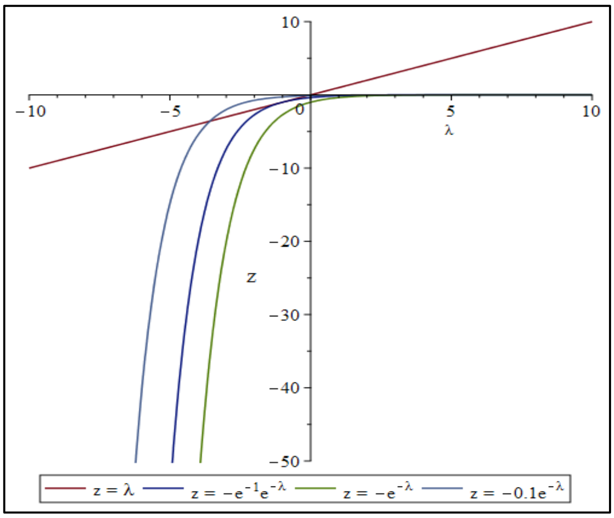

Figure 4: For real and negative eigenvalues

In Figure 3, the value of β is assumed that it is positive and equal 1, and there is one intersection point is obtained and located at a positive λ. Thus, the solution for $Eq.6$ is expected to grow exponentially and it is reasonable to conclude there is an unstable equilibrium at $U = 0$ due to the positive eigenvalue.

In Figure 4, the value of β is set in negative. Three possibilities were obtained based on the different values of β used which are zero, one, or two intersections for the two equations. There was one intersection point called single real root that happens when the curves are tangent of the $z = \lambda$. The corresponding value of β can be easily found out since there was the same slope for both curves and the straight line has a slope of 1. Thus,

$$\begin{aligned} \frac{d}{d\lambda}(\lambda) &= \frac{d}{d\lambda}(\beta e^{-\lambda}) \\ 1 &= -\beta e^{-\lambda} \\ e^{\lambda} &= -\beta \\ \beta &= -e^{\lambda} \end{aligned} \qquad Eq.14$$

After that, replace the equation found into the $z = \beta e^{-\lambda}$, and then

$$\begin{aligned} z &= (-e^{\lambda})e^{-\lambda} \\ z &= -1 \end{aligned}$$

Due to $z = \lambda$, it is known that when the $\lambda = -1$, the tangency happened and the β is equals to $-e^{-1}$. As the critical value for tangency was obtained, it can be used as refer, to determine the value of β corresponding to zero or two intersection points. In the graph, there are two intersection points given that the β is set with equal to -0.1 which was greater than $-e^{-1}$ and no exist of intersection point when assuming that equal to -1 which is smaller than $-e^{-1}$.

Based on the discussion above, several conditions are concluded for the $Eq.8$, the characteristic equation given that there are real and for both positive and negative eigenvalues, λ.

1. If $\alpha\tau > 0$, then there is exactly one real and positive eigenvalue, the equilibrium is unstable.
2. If $-e^{-1} < \alpha\tau < 0$, there are exactly two real negative eigenvalues, the associated exponential solution is decay to 0 over time. The equilibrium can be said as a stable equilibrium.
3. If $\alpha\tau = -e^{-1}$, then there is a single negative eigenvalue with value -1 and lead to a result of stable equilibrium.
4. If $\alpha\tau < -e^{-1}$, there are no real roots with exponentially decaying to the solution.

3.3 Logistic growth model and Hutchinson's equation

The second comparison was conducted by using Logistic growth model and Hutchinson's equation that introduced by [3] and [6] respectively, the model equations is written as following:

$$\frac{dp}{dt} = \alpha\left(1 - \frac{p(t)}{K}\right)p(t) \qquad Eq.15$$

$$\frac{dp}{dt} = \alpha\left(1 - \frac{p(t-\tau)}{K}\right)p(t) \qquad Eq.16$$

Similar procedure was repeated as the previous section that for exponential model which in order to make comparison and analyzing on both models.

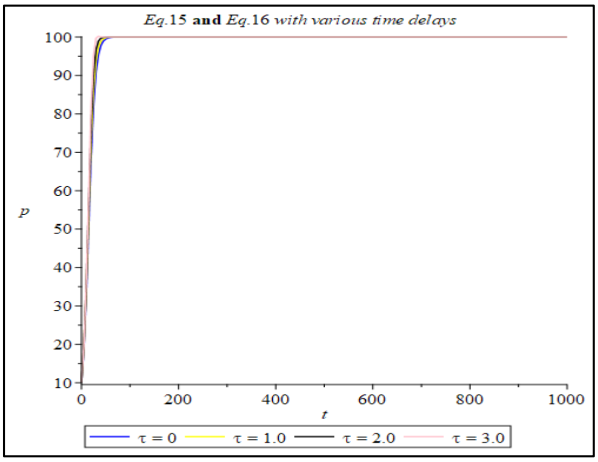

Figure 5: $Eq.15$ and $Eq.16$ with various time delays

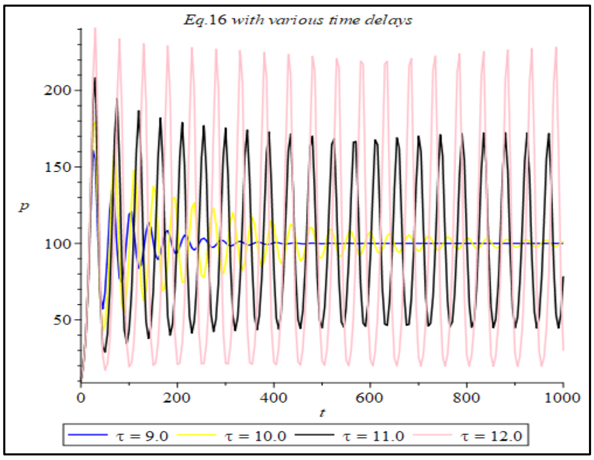

Figure 6: $Eq.16$ with various time delays

Based on the Figure 5, it can clearly be seen that the solutions for $Eq. 16$ which is the Hutchinson equation are almost similar with the $Eq. 15$ which is the logistic equation if the delay is sufficiently small given that the time delay values are equal to 1, 2 and 3. All the solutions shown above are growth and converge to the value of carrying capacity.

From the Figure 6, there was a small perturbation that occurs given the time delay is equal to 9 and 10, but it had converged back to the value of carrying capacity at the end over time which means the equilibrium until the current delay is still in stable. However, when the time delay was equal 11, there was an arise of oscillation or periodic solution over time for the equilibrium, which was the sign of Hopf-bifurcation. It is reasonable to say that there is a critical value in between 10 and 11 that leads to the bifurcation to occur. Naturally, the equilibrium that originally in steady state has become unstable. As the time delay increases, the periodic solution arose and led to the loss of stability for a stable equilibrium. This condition is similar as in exponential model.

In order to conduct the stability analysis, the dimensionless analysis was repeated for the Hutchinson's equation. Let introduce two equations in order to reduce the number of parameters of Hutchinson's equation by assuming $y = \frac{p}{K}$ and $t' = \frac{t}{\tau}$, and the following equation will be achieved

$$\begin{aligned} \frac{d(Ky)}{d(\tau t')} &= \alpha\left(1 - \frac{Ky(\tau t' - \tau)}{K}\right)(Ky) \\ \frac{d(y)}{d(t')} &= (\tau)\alpha y\big(1 - y(t'-1)\big) \\ \frac{dy}{dt'} &= \tau\alpha y[1 - y(t'-1)] \end{aligned}$$

Let $\sigma = \tau\alpha$, then the interchange of dimension is completed

$$\frac{dy}{dt'} = \sigma y[1 - y(t'-1)] \qquad Eq. 17$$

The equilibrium for both $Eq. 15$ and 16 is similar due to the value of function $p(t-\tau)$ is totally identical with that of $p(t)$ at equilibrium since the solution does not change with the moment of t in the steady state. Therefore, only the $Eq. 17$ will be investigated and the equilibrium found can be applies for $Eq. 15$ given that if it is also in dimensionless form.

$$\begin{aligned} 0 &= \sigma y[1-y] \\ \sigma y = 0 \quad &\Big| \quad 1 - y = 0 \\ y = 0 \quad &\Big| \quad y = 1 \end{aligned}$$

As the result shown, there are two equilibriums for both equations which are in dimensionless form, they are 0 and 1.

3.4 Stability Test: Logistic growth model and Hutchinson's equation

It should be known that there will be no changes of stability for the logistic growth model and the solution of the logistic growth model was mainly depending on the value of α. The solution will grow and move toward to the equilibrium K when α is positive, the solution was approaching the other equilibrium 0 when α is negative. Given that the initial value of value was greater than the carrying capacity, the solution was also moving toward to the equilibrium K as well. It should be notice that the equilibrium K and 0 are valid for the dimensional form but not for dimensionless form, the equilibrium for dimensionless form were 1 and 0 that obtained in previous analysis.

For the stability analysis of Hutchinson's equation, the perturbation from each equilibrium was introduced to see whether the solution returns to the steady state and the method used is called linearization. The equilibriums investigated were 0 and 1 given that the Hutchinson's equation was in dimensionless form.

First, the perturbations z from $y = 0$ were satisfying the linear equation $\frac{dz}{dt'} = \sigma z$, which was differentiation function with exponential growth and decay. This means that the equilibrium $y = 0$ was unstable. For equilibrium $y = 1$, an equation $y = z + 1$ was introduced in order to analyze the state stability.

$$\frac{dz}{dt'} = \sigma(z+1)[1 - \{z(t'-1) + 1\}]$$
$$\frac{dz}{dt'} = -\sigma(z+1)z(t'-1)$$

For small perturbation, given that $z + 1 \approx 1$, then the linearized equation is

$$\frac{dz}{dt'} = -\sigma z(t'-1) \qquad Eq. 18$$

For the $Eq. 18$ stated above, it should not stranger to it since it actually was similar to the $Eq. 7$ that investigated the exponential equation with delay. Therefore, the analysis result for the stability for $Eq. 7$ can be applied for $Eq. 18$.

In this case, there was a negative sign for $Eq. 18$ but $Eq. 7$ not, which is $\beta = -\sigma$. Thus, some modification will be taken and the critical value for Hopf-bifurcation will be change as well. Remember that $\sigma = \tau\alpha$.

1. If $-\frac{\pi}{2} < -\sigma < 0$, which is equal to $0 < \tau\alpha < \frac{\pi}{2}$, the equilibrium $U = 0$ is stable due to the real part is smaller that 0 for all eigenvalues.
2. If $\tau\alpha = \frac{\pi}{2}$, then the real part is equal to 0 and there are exist of purely complex conjugate eigenvalues, $\lambda = \pm i\frac{\pi}{2}$. Hopf-bifurcation occurs.
3. If $-\sigma < -\frac{\pi}{2}$, which is $\tau\alpha > \frac{\pi}{2}$, then there are eigenvalues with real parts greater than 0 and the equilibrium $U = 0$ is unstable.

From the statement above, it can be said that the small perturbation from equilibrium 1, was decaying to 0 at the end over time, and conclude that the corresponding equilibrium was stable. At $\sigma = \frac{\pi}{2}$, there were purely complex conjugate eigenvalues, and induce that the loss of stability of equilibrium 1. Start from the point, the periodic solution has risen and satisfied the condition of Hopf bifurcation.

Given that there are real eigenvalues for the characteristic equation of $Eq. 18$,

1. If $\sigma < 0$, then there is exactly one real and positive eigenvalue, the equilibrium is unstable.
2. If $0 < \sigma < e^{-1}$, there are exactly two real negative eigenvalues, the associated exponential solution is decay to 0 over time. The equilibrium can be said as a stable equilibrium.
3. If $\sigma = e^{-1}$, then there is a single negative eigenvalue with value -1 and lead to a result of stable equilibrium.
4. If $\sigma > e^{-1}$, there are no real roots with exponentially decaying to the solution.

4. Conclusion

Based on the result obtained in the study, it can be concluded that there is no significant difference for both ODE and DDE population dynamics models given that the value of time delay is small enough. However, there must an existent of time delay for all processes and systems but not respond instantaneous, so that the delayed population dynamics models are recommended which can present a better approximation and analysis based on the real-life phenomenon.

With the basic theory of DDE, as the time delay gets larger, there is something that happens on the trajectories of the model solution. The equilibrium which is initial in a stable state becomes unstable, this leads to the condition of the solution will not converge at one state level over time. As the loss of stability, the periodic solutions occur and oscillate over time. When the time delay keeps increasing, the solution will become more and more non-stationary. Thus, the critical value that leads to this phenomenon is necessary to be obtained.

For the exponential decay model, if the product of the intrinsic growth rate and the time delay, $\alpha\tau$ is equal to $-\frac{\pi}{2}$, then the Hopf-bifurcation will occur. Strat from that point, the equilibrium will be unstable and there is a periodic solution. In Hutchinson's equation, it was tested as well to determine the critical value for the happening of Hopf-bifurcation as well. The condition for the happening of Hopf-bifurcation phenomenon is totally opposite with the discussion on the exponential decay model. The non-trivial equilibrium will be stable if the product of the intrinsic growth rate and the time delay was smaller than $\frac{\pi}{2}$ due to the negative real part for all eigenvalues. As the product becomes larger, a critical value will be met, which is $\frac{\pi}{2}$, then the similar phenomenon as in the exponential decay model will be repeated with the risen of periodic solution and loss of stability.

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References

[1] F. Brauer, and C. Castillo-Chavez, "Mathematical Models in Populations Biology and Epidemiology," Springer, vol. 2, no. 40, 2012.

[2] T.R. Malthus, "An Essay on the Principle of Population Growth," Dover Publication Inc, 1798.

[3] P.F. Verhulst, "Notice sur la loi que la population suit dans son accroissement," Corr. Math. Phy, no. 10, pp. 113-121, 1936.

[4] K. Yang, "Delay Differential Equations: With Applications in Population Dynamics," Aca. Press, Inc, vol. 191, 1993.

[5] N.K. Vaidya. and J. Wu, "Modeling Spruce Budworm Population Revisited: Impact of Physiological Structure on Outbreak Control," Bulletin of Mathematical Biology, vol. 70, no. 3 pp. 760-784, 2008.

[6] G.E. Hutchinson, "Circular Casual Systems in Ecology," Annals of the N.Y. Aca. of Sci., vol. 50, pp. 221-246, 1948.

[7] T. Freitas, "Modeling Population Dynamics with Ordinary and Delay Differential Equations," 2015.

[8] E. Hopf, "Bifurcation of a Periodic Solution from a Stationary Solution of a System of Differential Equations," no. 5, pp. 163-205, 1942.

[9] T. Faria. and L. Magalhães, "Normal Forms for Retarded Functional Differential Equations with Parameters and Applications to Hopf Bifurcation," J. Diff. Equation, vol. 122, no. 2, pp. 181-200, 1995.