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Laplace Transformation for Control and Analysis of Nonlinear System

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Abstract: Nonlinear systems play an important role in modelling real-world problems. In this paper, the Laplace transform is applied to control and analyze nonlinear systems. For this purpose, the nonlinear system is linearized and converted into a system of algebraic equations by applying the Laplace transform. Then, the Routh-Hurwitz criterion is performed for stability analysis. In addition, the transfer function can be expressed in a partial fraction form to provide the solution of the system when the inverse Laplace transform is used. For illustration, the Vidale-Wolfe advertising model was studied. The results showed that the system is stable. The graphical solutions revealed that the different advertising strategies would provide different market shares, where the respective performance index was enclosed. In conclusion, the application of the Laplace transformation to nonlinear systems is properly presented.

Keywords: Laplace Transformation, Transfer Function, Stability Analysis, Routh-Hurwitz Criterion, Vidale-Wolfe Advertising Model

1. Introduction

Many real-world problems are modelled in a nonlinear system since they are inherently nonlinear in nature [1]. According to [2], a nonlinear system is a system of nonlinear equations, which is comprised of equations of algebraic, differential, partial differential, functional, integral or a combination of these equations. Nonlinear systems show the complex relationship among the variables in the system. Since the applications of nonlinear systems have been well-defined in the literature, the importance of the study of nonlinear systems provides a significant contribution to communities and societies. However, solving a nonlinear system for the solution is not easy and costly, even control and analysis of the nonlinear system would not be performed systematically [3]. Therefore, applying a simple computational technique to study nonlinear systems gives the motivation of using the Laplace transform for modeling and control nonlinear systems [4]. Laplace transform can be used to obtain the solution of the nonlinear system without solving the system of differential equation directly when the linearization is applied to the nonlinear system [5]. In fact, the Laplace transform converts the system

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of differential equations into a system of algebraic equations in obtaining the system solution. The calculation procedure in the Laplace transform simplifies the complicated steps and provides a practical way in the mathematical modelling of initial value problems.

In this paper, the transfer function is defined after using the Laplace transform to the nonlinear system. The characteristic equation [6], which is the polynomial in the transfer function, is referred to construct the Routh-Hurwitz array. Here, the Routh-Hurwitz criterion [7], which examines the stability of a system, is performed. It is mentioned that a system is stable if and only if there have no changes of sign in the first column in the array. From this point of view, the number of sign changes in the first column is equal to the number of the roots that lie on the right-side plane [8]. On the other hand, the transfer function can be expressed in a partial fraction form and applying the inverse Laplace transform, the system solution would be obtained. For illustration, a simple Vidale-Wolfe advertising model is studied.

The rest of the paper is organized as follows. In Section 2, the problem is described, and the Laplace transform is discussed to control and analyze the nonlinear system. In Section 3, an illustrative example of the Vidale-Wolfe advertising model is studied. The Routh-Hurwitz criterion is applied to examine the stability of the advertising system and the different control efforts are considered to investigate the effect to the final market share. Finally, a conclusion is made.

2. Materials and Methods

Consider a general class of the nonlinear dynamical system [9] given by

$$\dot{x}(t) = f(x(t), u(t)), \quad \text{Eq. 1}$$

$$y(t) = h(x(t), u(t)), \quad \text{Eq. 2}$$

with $x(0) = x_0$ and $t \geq 0$, where $x \in \Re^n$ is the state variable, $u \in \Re^m$ is the control input and $y \in \Re^p$ is the output vector, whereas $f : \Re^n \times \Re^m \to \Re^n$ represents the plant dynamics. Here, the aim is to determine the admissible control input such that the following performance index [10]

$$J(u) = \phi(x(T)) + \int_0^T L(x(t), u(t))dt \quad \text{Eq. 3}$$

is minimized over the dynamical system that is given in Eq. 1 and Eq. 2, where $\phi : \Re^n \to \Re$ is the terminal cost, $L : \Re^n \times \Re^m \to \Re$ is the operating cost, J is the scalar function and T is the final time.

2.1 Linearization of Nonlinear System

Applying the Taylor series expansion [9] to linearize the state equation, that is,

$$f(x,u) \approx f(x_0, u_0) + \frac{\partial f(x_0, u_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0, u_0)}{\partial u}(u - u_0), \quad \text{Eq. 4}$$

where x_0 and u_0 are the equilibrium points for the state and control variables, respectively. At the steady state condition, it is noticed that $f(x_0, u_0) = 0$. Therefore, the system in Eq. 1 and Eq. 2 can be written by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad \text{Eq. 5}$$

$$y(t) = Cx(t) + Du(t), \quad \text{Eq. 6}$$

where A, B, C and D are the system matrices. While, the performance index in Eq. 3 is approximated in the quadratic criterion [10] given by

$$J(u) = \frac{1}{2}x(T)^{\mathrm{T}}S(N)x(T) + \frac{1}{2}\int_0^T (x(t)^{\mathrm{T}}Qx(t) + u(t)^{\mathrm{T}}Ru(t))dt \qquad \text{Eq. 7}$$

where $S(N)$, Q and R are the weighting matrices.

2.2 Laplace Transform and Transfer Function

Taking the Laplace transform

$$\mathrm{L}\{x(t)\} = \int_0^\infty x(t)e^{-s\cdot t}dt\,, \qquad \text{Eq. 8}$$

the linearized systems in Eq. 5 and Eq. 6 are written by

$$sX(s) - x(0) = AX(s) + BU(s)\,, \qquad \text{Eq. 9}$$

$$Y(s) = CX(s) + DU(s)\,, \qquad \text{Eq. 10}$$

where $X(s) = \mathcal{L}\{x(t)\}$, $U(s) = \mathcal{L}\{u(t)\}$, $Y(s) = \mathcal{L}\{y(t)\}$ and $\mathcal{L}\{\dot{x}(t)\} = sX(s) - x(0)$.

After some algebraic manipulations, Eq. 9 becomes

$$X(s) = (sI - A)^{-1}(BU(s) + x(0))\,. \qquad \text{Eq. 11}$$

Substitute Eq. 11 into Eq. 10 to give

$$Y(s) = (C(sI - A)^{-1}B + D)U(s) + C(sI - A)^{-1}x(0)\,. \qquad \text{Eq. 12}$$

Then, the ratio of the output to the input of the system is given by

$$G(s) = \frac{Y(s)}{U(s)} = (C(sI - A)^{-1}B + D)\,, \qquad \text{Eq. 13}$$

for the initial state $x(0) = 0$. Hence, Eq. 13 is termed as the transfer function [11] of the system in Eq. 1 and Eq. 2.

Mathematically, the transfer function in Eq. 13 is defined as

$$G(s) = \frac{p(s)}{q(s)} \qquad \text{Eq. 14}$$

where

$$p(s) = p_{n-1}s^{n-1} + p_{n-2}s^{n-2} + \cdots + p_0 \qquad \text{Eq. 15}$$

$$q(s) = q_n s^n + q_{n-1}s^{n-1} + q_{n-2}s^{n-2} + \cdots + q_0 \qquad \text{Eq. 16}$$

are the polynomial with coefficients $p_{n-1}, p_{n-2}, \cdots, p_0$ and $q_n, q_{n-1}, q_{n-2}, \cdots, q_0$, respectively.

Consequently, by taking the inverse Laplace transform to Eq. 11 and Eq. 12, the system solution is obtained from

$$x(t) = \mathrm{L}^{-1}\{X(s)\}\,.$$

2.3 Routh-Hurwitz Criterion Implementation

Refer to the transfer function in Eq. 14, the polynomial in Eq. 16 is written as the characteristic equation $q(s)=0$. The coefficients of the characteristic equation are filled into an array as shown in Table 1.

Table 1: Initial array

Item	Element				
s^n	q_n	q_{n-2}	q_{n-4}	q_{n-6}	$\cdots$
s^{n-1}	q_{n-1}	q_{n-3}	q_{n-5}	q_{n-7}	$\cdots$

The further rows are then listed in Table 2.

Table 2: Routh-Hurwitz array

Item	Element				
s^n	q_n	q_{n-2}	q_{n-4}	q_{n-6}	$\cdots$
s^{n-1}	q_{n-1}	q_{n-3}	q_{n-5}	q_{n-7}	$\cdots$
s^{n-2}	a_1	a_2	a_3	a_4	$\cdots$
s^{n-3}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-4}	c_1	c_2	c_3	c_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^0	z_1	z_2	z_3	z_4	$\cdots$

The elements in Table 2 are calculated from [12]

$$a_1=\frac{q_{n-1}q_{n-2}-q_nq_{n-3}}{q_{n-1}},\ a_2=\frac{q_{n-1}q_{n-4}-q_nq_{n-5}}{q_{n-1}},\ a_3=\frac{q_{n-1}q_{n-6}-q_nq_{n-7}}{q_{n-1}} \quad \text{Eq. 17}$$

and so on,

$$b_1=\frac{a_1q_{n-3}-a_2q_{n-1}}{a_1},\ b_2=\frac{a_1q_{n-5}-a_3q_{n-1}}{a_1},\ b_3=\frac{a_1q_{n-7}-a_4q_{n-1}}{a_1} \quad \text{Eq. 18}$$

and so on,

$$c_1=\frac{b_1a_2-b_2a_1}{b_1},\ c_2=\frac{b_1a_3-b_3a_1}{b_1},\ c_3=\frac{b_1a_4-b_4a_1}{b_1} \quad \text{Eq. 19}$$

and so on.

According to the Routh-Hurwitz criterion [12], a system is said to be stable if and only if there has the same signs in the first column in the array. For a system to be stable, this criterion is both necessary and sufficient. In addition, the number of roots with positive real parts is equal to the number of changes in sign of the first column in the array.

3. Results and Discussion

For illustration, the Vidale-Wolfe advertising model [13] is studied, where the relationship between the advertising and sales is observed. Consider the following model,

$$\dot{x}(t)=r\cdot u(t)\cdot(1-x(t))-\lambda\cdot x(t),\quad x(0)=x_0, \quad \text{Eq. 20}$$

where $\dot{x}(t)$ is the rate of sales, r is the response to advertising effort constant, $u(t)$ is the advertising budget, λ is the sales exponential decay constant and $x(t)$ is market share with an initial constant value x_0.

The linearized model of the Vidale-Wolfe advertising model is defined by

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad \text{Eq. 21}$$

with $A = -r \cdot u_0 - \lambda$, $B = r \cdot (1 - x_0)$, where u_0 is the initial control effort and the market share is measured by

$$y(t) = x(t). \qquad \text{Eq. 22}$$

Consider the initial condition and parameters [14],

$$r = 1.2, \; \lambda = 0.2, \; x_0 = 0.1, \; u_0 = 10,$$

the transfer function is obtained as follows,

$$G(s) = \frac{27}{5(5s + 61)}. \qquad \text{Eq. 23}$$

The characteristic equation is

$$q(s) = s + 12.2. \qquad \text{Eq. 24}$$

The elements in the first column of the Routh-Hurwitz array are shown in Table 3.

Table 3: Elements in first column

Item	Element
s^1	1
s^0	12.2

Since there have no sign changes in the first column of the array, it is concluded that the model is stable. Moreover, the stability of the model is also verified by the pole and eigenvalue of the model given as follows,

$$pole = -12.2, \; eigenvalue = -12.2.$$

Different control functions are selected to investigate the effective advertising effort on the market share. These control functions are

(a) Linear function,

$$u(t) = 10 - 2t \qquad \text{Eq. 25}$$

(b) Trigonometric function,

$$u(t) = 10 - \sin(t) \qquad \text{Eq. 26}$$

(c) Heaviside function,

$$u(t) = 10 - 6u_{0.3}(t) - 4u_{0.8}(t) \qquad \text{Eq. 27}$$

For a one-year period, the performance index of the model, which is the total profit, is maximized in the quadratic criterion, given by

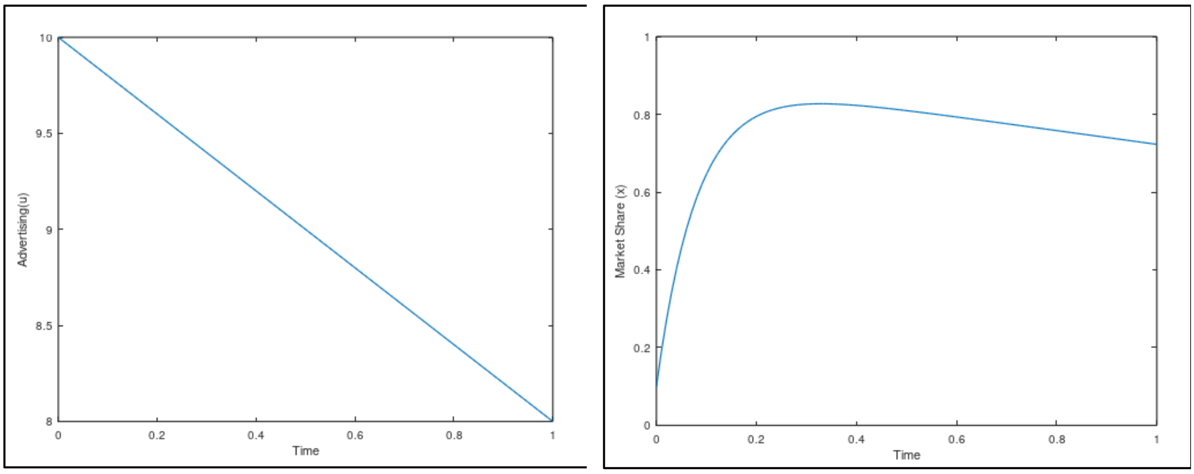
$$J(u) = \frac{1}{2}\int_0^1 ((x(t))^2 + (u(t))^2)dt \,. \qquad \text{Eq. 28}$$

The computation result is shown in Table 4, where J^* is the optimal value of the performance index for the respective control function used. Among the control functions, using the trigonometric function as the control effort would give the maximum total profit.

Table 4: Optimal performance index

Control Function $u(t)$	Performance Index J^*
Linear function	40.954
Trigonometric function	45.860
Heaviside function	19.127

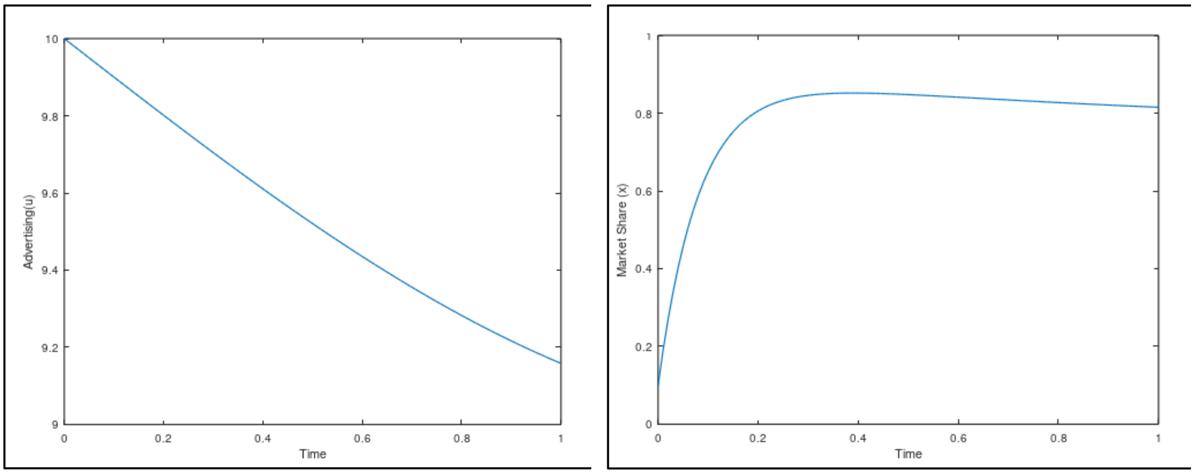
The graphical results for the different control efforts and the relative market share are shown in Figures 1, 2 and 3. In Figure 1 (a), the company initially spends the maximum advertising effort at $u_0 = u_{\max} = 10$ and takes the linear control effort $u(t) = 10 - 2t$ to reach the optimal market share. To increase the profit, the advertising effort decreases sharply until $u = 0$ because the decay of the market share is slow. Figure 1 (b) shows the initial market share at $x_0 = 0.1$ units and the final market share at $x_T = 0.7227$ units.



(a) (b)

Figure 1: (a) Linear advertising effort and (b) Market share with linear control function

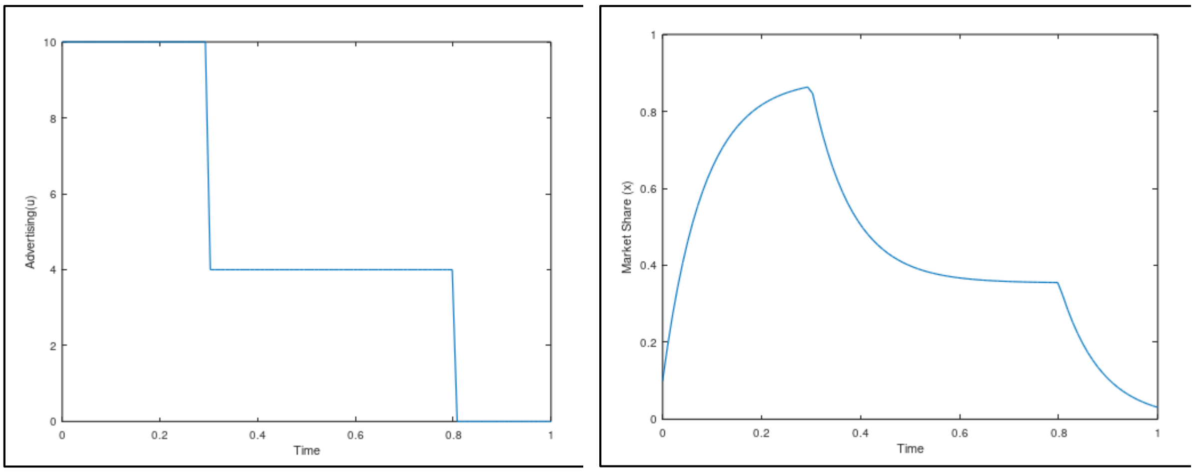
In Figure 2 (a), the company initially spends the maximum advertising effort at $u_0 = u_{\max} = 10$ and considers to use the trigonometric function $u(t) = 10 - \sin(t)$ to reach the optimal market share. To increase the rate of sales, the advertising effort decreases slowly until $u = 9.1585$ since there is only a slightly decay occurred in the market share. Figure 2 (b) shows the initial market share at $x_0 = 0.1$ units and the final market share at $x_T = 0.8151$ units.



(a) (b)

Figure 2: (a) Trigonometric advertising effort and (b) Market share with trigonometric control function

In Figure 3(a), the company initially spends the maximum advertising effort at $u_0 = u_{\max} = 10$ and use the Heaviside function $u(t) = 10 - 6u_{0.3}(t) - 4u_{0.8}(t)$ as the control effort to reach the optimal market share. The advertising effort reduces to $u = 4$ for maintaining the equilibrium. In the end, advertising effort reduces to $u = 0$ since there is only a slightly decay occurred in the market share. Figure 3 (b) shows the initial market share at $x_0 = 0.1$ units, and the final market share at $x_T = 0.0310$ units.



(a) (b)

Figure 3: (a) Heaviside advertising effort and (b) Market share with Heaviside control function

Notice that the final state of the market share is different. This reveals that the effectiveness of the advertising effort within the one-year period is contributed by using the trigonometric control function, which gives the maximum final market share of 81.5% with the maximum total profit of 45.860 units. Thus, from the graphical results and performance index, it is clearly shown that the advertising effort will affect the market share. So, the advertising decision could be suggested that if there is a sudden decrease of advertising effort, the market share will be decreased apparently. If the advertising effort decreases slightly in a time range, the market share can reach equilibrium and then decays slightly.

4. Conclusion

In this paper, the use of the Laplace transform to control and analyze the nonlinear system was discussed. After doing the linearization, the transfer function, which is resulted from using the Laplace transform, provided a useful practical approach in handling the nonlinear system since the system solution could be obtained from solving the system of algebraic equations. From the transfer function, the characteristic equation was used to construct the Routh-Hurwitz array and the stability of the system was examined. From the illustrative example, the Vidale-Wolfe advertising model was studied with the different control efforts. The results showed the applicability of using the Laplace transform in managing the nonlinear system. In conclusion, the Laplace transform is one of the efficient computational tools for solving nonlinear systems. It is recommended that an efficient nonlinear control strategy could be applied to study the complex advertising model in the future.

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