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Chebyshev Tau Method to Solve Linear Klein Gordon Equation

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Abstract In this study, Chebyshev Tau Method is used to solve the linear system of Linear Klein Gordon equations. Maple software is used to digitally solve the problem. Chebyshev-Tau method prove to be accurate to find Klein Gordon equation. Suggestion future work would be solving the nonlinear Klein Gordon using the same method.

Keywords: Chebyshev Tau Method, Shifted Chebyshev Polynomial, Tau Method, Klein Gordon Equation

1. Introduction

The linear partial differential equations arising in physic and engineering play important role in mathematical modeling. Searching the exact solutions to these linear or nonlinear models gains importance. Regarding the above reason, a lot of methods were developed to solve the differential equation [1].

C. Lanczos introduced the use of Chebyshev polynomial in 1938 in order to find an approximate solution for a physical problem. C .Lanczos remarked that truncation of the series solution of a differential equation, in some way, equivalent to introducing a perturbation term in the right-hand side of the equation.The approach of the Tau method is based on three simple operational matrices.that been introduced for the first time in 1981 by E.l Ortiz and H.Samara for the numerical solution of nonlinear ordinary differential equation and in 2002 it was extended by M.Hosseini and S.Shahmorad for the numerical solution of linear integrodifferential equation[4]. In this study, linear Klein Gordon is defined as below,

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$$\frac{\partial^2 \mathrm{u}}{\partial t^2} - \frac{\partial^2 \mathrm{u}}{\partial x^2} + f(x,t) = 0, \quad 0 < x < l \ , \quad 0 < \mathrm{t} < \tau \qquad Eq.1$$

With initial condition

$$u\ (x, 0) = g_1(x), \qquad 0 < x < l\ , \qquad Eq.2$$

$$u(x, 0) = g_2(x), \qquad 0 < x < l\ , \qquad Eq.3$$

And initial boundary condition

$$u(0, t) = h_1(t), \qquad 0 < t \leq \tau \qquad Eq.4$$

$$u_t(l, t) = h_1(t), \qquad 0 < t \leq \tau \qquad Eq.5$$

The use of orthogonal polynomials in numerical analysis and approximation theory has been greatly expanded in the last decades. Indeed, all kinds of Chebyshev polynomials received considerable attention in the numerical solution of ordinary, partial, and FDEs [7].

The aim of this paper is to apply Chebyshev Tau Method to obtain the approximate solutions of linear Klein Gordon equations. We demonstrate the accuracy and efficiency of the Chebyshev Tau Method through some test examples. Numerical comparison will be made against the actual result from numerical calculation.

2. Methodology

Shifted Chebyshev Polynomial

Define Chebyshev polynomial $T_n(\bar{x})$ as follows (szego, 1975)

$$T_n(\bar{x}) = \cos\left(n\, cos^{-1}(\bar{x})\right), -1 \leq \bar{x} \leq 1 \qquad Eq.6$$

Define Independent variable x between -1 and 1. To solve (1.1) to (1.4), transform the domain into values between 0 and h.

Let

$$x = \frac{h}{2}(1 + \bar{x}) \qquad Eq.7$$

We defined The shifted Chebyshev polynomials of x as

$$T_0^h(x) = 1, \qquad i = 1,2,\ldots \qquad Eq.8$$

$$T_1^h(x) = \frac{2x}{h} - 1, \qquad Eq.9$$

$$T_{i+1}^h(x) = \left(2 - \frac{4x}{h}\right) T_i^h(x) - T_{i-1}^h(x) \qquad Eq.10$$

The orthogonal condition of the shifted Chebyshev polynomials is given by

$$\int_0^h \frac{T_i^h(x) T_j^h(x)}{\sqrt{hx - x^2}} dx = \begin{cases} 0 & i \neq j \\ \frac{\pi}{2} & i = j \neq 0 \\ \pi & i = j = 0. \end{cases} \qquad Eq.11$$

Approximate the shifted Chebyshev polynomials by function y(x) as

$$y_m(x) = \sum_{j=0}^{m-1} c_j T_j^h(x) = C^T \Phi_{m,h}(x), \qquad Eq.12$$

Superscript T was denoted as transpose, while C is shifted Chebyshev polynomials vector and $\Phi_{m,h(x)}$ is the shifted Chebyshev polynomials vector.

These vectors are given as:

$$C = [c_0, c_1, \ldots, c_{m-1}]^T \qquad Eq.13$$

and

$$\Phi_{m,h}(\mathrm{x}) = [T_0^h(x), T_1^h(x), \ldots., T_{m-1}^h(x)]^T \qquad Eq.14$$

2.2 Methods

Integrate Eq. 1 from 0 to t and using equations Eq. 2, Eq. 3, Eq. 4, and Eq. 5 we will obtain:

$$u(x,t) - g_1(x) - t g_2(x) - \int_0^t \int_0^t u_{(x,t)}(x,t) dt dt + \int_0^t \int_0^t f(x,t) dt dt = 0 \qquad Eq.15$$

Function $u(x,t)$can be expended in term of double shifted Chebyshev polynomial as follow

$$u_{m,n}(x,t) = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} a_{ij} T_i^{\tau}(t) T_j^{\mathrm{l}}(x) = \Phi_{n,\tau}^T(t) A \Phi_{m,l}(x) \qquad Eq.16$$

where A is define as,

$$\mathrm{A}=\begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0,m-1} \\ a_{10} & a_{11} & \cdots & a_{1,m-1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n-1,0} & a_{n-1,1} & \cdots & a_{n-1,m-1} \end{pmatrix}, \qquad Eq.17$$

We define the general operational matrix of integration as

$$\int_0^t \int_0^t \dots \int_0^t A\Phi_{n,\tau}(t)(dt)^k = P^k \Phi_{n,\tau}(x) \qquad Eq.18$$

From property of shifted Chebyshev polynomial in [2]

$$\frac{d^k \Phi_{m,l}(x)}{d^k} = D^k \Phi_{m,l}(x) \qquad \text{Eq.}19$$

where D is

$$\mathrm{D}=\frac{1}{l}\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ -2 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & -8 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ -6 & 0 & -12 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & -16 & 0 & -16 & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (m-2)[(-1)^{m-2}-1] & -2(m-2)[(-1)^{m-2}+1] & 2(m-2)[(-1)^{m-2}-1] & -2(m-2)[(-1)^{m-2}+1] & \cdots & \cdots & 0 & 0 \\ (m-2)[(-1)^{m-1}-1] & -2(m-1)[(-1)^{m-1}+1] & 2(m-1)[(-1)^{m-1}-1] & -2(m-1)[(-1)^{m-1}+1] & \cdots & \cdots & -4(m-1) & 0 \end{pmatrix} \qquad Eq.20$$

Next $g_{1m}(x)$ and $g_{1m}(x)$ for $0 < x < l$ and $0 < \mathrm{t} < \tau$ can be approximated as

$$g_{1m}(x) = \Phi_{n,\tau}^T(t) E \Phi_{m,l}(x), \qquad Eq.21$$

$$t g_{2m}(x) = \Phi_{n,\tau}^T(t) H \Phi_{m,l}(x) \qquad Eq.22$$

where E and H define as,

$$E=\begin{pmatrix} a_{10} & a_{11} & \cdots & a_{1,m-1} \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}, \qquad Eq.23$$

$$H=\frac{\tau}{2}\begin{pmatrix} g_{20} & g_{21} & \cdots & g_{2,m-1} \\ -g_{20} & -g_{21} & \ldots & -g_{2,m-1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix} \qquad Eq.24$$

We approximate $f(x,t)$ as:

$$f_{(n,m)}(x,t) = \Phi_{n,\tau}^{T}(t)F\Phi_{m,l}(x) \qquad Eq.25$$

where F is n × m matrix equation

Substituting Eq. 16 and Eq. 17 we have

$$\int_0^t \int_0^t \ldots \int_0^t u_{n,m}(x,t)(dt)^k = \Phi_{n,\tau}^{T}(t)(P^T)A\Phi_{m,l}(x) \qquad Eq.26$$

where P is,

$$P = \tau\begin{pmatrix} \frac{1}{2} & \frac{-1}{2} & 0 & \ldots & 0 & 0 & 0 \\ \frac{1}{8} & 0 & \frac{-1}{8} & \ldots & 0 & 0 & 0 \\ \frac{-1}{6} & \frac{1}{4} & 0 & \ldots & 0 & 0 & 0 \\ \frac{-1}{16} & 0 & \frac{1}{8} & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{-1}{2(n-1)(n-3)} & 0 & 0 & \ldots & \frac{1}{4(n-3)} & 0 & \frac{-1}{4(n-1)} \\ \frac{-1}{2n(n-2)} & 0 & 0 & \ldots & 0 & \frac{1}{4(n-2)} & 0 \end{pmatrix}, \qquad Eq.27$$

derive from

$$\int_0^t \int_0^t \ldots \int_0^t \Phi_{n,\tau}(t)(dt)^k = P^k\Phi_{n,\tau}(t). \qquad Eq.28$$

We obtain the general integration form of $f(x,t)$ from Eq. 27.

$$\int_0^t \int_0^t \dots \int_0^t f_{n,m}(x,t)(dt)^k = \Phi_{n,\tau}^T(t)(P^T)F\Phi_{m,l}(x) \qquad Eq.29$$

Using equations Eq. 16, Eq. 19 and Eq. 28 we obtain

$$\int_0^t \int_0^t \dots \int_0^t \frac{\partial u_{(x,t)}(x,t)}{\partial x^p}(dt)^k = \Phi_{n,\tau}^T(t)(P^T)^k AD^p \Phi_{m,l}(x) \qquad Eq.30$$

Applying Eq. 16, Eq. 21, Eq.22, Eq. 25 to Eq. 30. The residual $R_{n,m}(x,t)$for equation Eq. 15 can be given by

$$R_{n,m}(x,t) = \Phi_{n,\tau}^T(t)[Q]\Phi_{m,l}(x) = \Phi_{n,\tau}^T(t)(P^T)F\Phi_{m,l}(x) \qquad Eq.31$$

where

$$Q = A - E - H - (P^T)AD^2 + (P^T)^2F \qquad Eq.32$$

Generate n× (m-2) linear equation by using the following system:

$$Q_{i,j} = 0, \qquad Eq.33$$

$i = 0,1, \dots, n-1,$

$j = 0,1, \dots., m-3.$

Substitute equation Eq.16 into equation Eq.4 and Eq.1 have

$$\Phi_{n,\tau}^T(t)A\Phi_{m,l}(0)=h_1(t) \qquad Eq.34$$

$$\Phi_{n,\tau}^T(t)A\Phi_{m,l}(l) = h_2(t) \qquad Eq.35$$

Equation Eq. 34 and Eq.35 are collocated at n points. We use shifted Chebyshev roots $t_i, i = 1, \dots, n\ of\ T_n^\tau$(t)

For these collocation points. The number of coefficient a_{ij}equal to n×m can be computed Substituting equation Eq. 33 to Eq. 35 we get the exact solution of

$$u_{m,n}(x,t) = \Phi_{n,\tau}^T(t)A\Phi_{m,l}(x) \qquad Eq.36$$

3. Results and Discussion

For this research,Chebyshev-Tau method will be focus on solving the linear Klein Gordon equation.The example will be solve to obtain the solutions.Next, the solution obtained from Maple will be compare to the exact solution of the problem.The result obtained demonstrate the efficiency and accuracy of Chebyshev-Tau method to exact solution.

Example 1

Consider the following linear Klein Gordon equation

$$\frac{\partial^2 \mathrm{u}}{\partial t^2} - \frac{\partial^2 \mathrm{u}}{\partial x^2} + t^2 - x^2 = 0 \qquad Eq.37$$

with initial conditions

$$u\,(x,0) = 0 \qquad 0 < x < 1\,, \qquad Eq.38$$

$$u(x,0) = 0 \qquad 0 < x < 1\,, \qquad Eq.39$$

and dirichlet boundary condition

$$u(0,t) = 0, \qquad 0 < t \leq 1 \qquad Eq.40$$

$$u(l,t) = \frac{1}{2}t^2, \qquad 0 < t \leq 1 \qquad Eq.41$$

Solution of eq.37 is

$$u_{3,3}(x,t) = \frac{1}{2}x^2t^2 \qquad Eq.42$$

Table 3.1: Estimated and exact value of u(x, t) for example 1.

x	exact	Chebyshev Tau	Absolute error
0.1	.00500	.00736	.00236
0.2	.02000	.02111	.00111
0.3	.04500	.04402	.00098
0.4	.08000	.07611	.00389
0.5	.12500	.11736	.00764
0.6	.18000	.16778	.01222
0.7	.24500	.23635	.00865
0.8	.32000	.30111	.01889
0.9	.40500	.39142	.01358
1.0	.50000	.48972	.01028

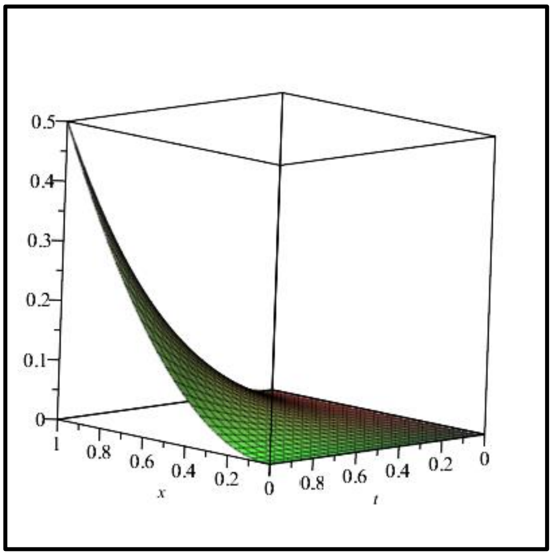

Figure 1: 3D graph of numerical solution for example 1

Based on Table 3.1, the Maple calculation gives almost the same answer approximately at 0.01 decimal places. This shows that the Chebyshev-Tau method is efficient and accurate to the exact solution for this example.

Example 2

$$\frac{\partial^2 \mathrm{u}}{\partial t^2} - \frac{\partial^2 \mathrm{u}}{\partial x^2} + t^2 \sin(x) + \sin(x) + t = 0 \qquad Eq.43$$

with initial conditions

$$u\,(x, 0) = 0 \qquad 0 < x < 1\,, \qquad Eq.44$$

$$u(x, 0) = 0 \qquad 0 < x < 1\,, \qquad Eq.45$$

and dirichlet boundary condition

$$u(0, t) = 0, \qquad 0 < t \leq 1 \qquad Eq.46$$

$$u(l, t) = 0, \qquad 0 < t \leq 1 \qquad Eq.47$$

Solution for Eq. 43 is

$$u_{3,3}(x, t) = \sin(x) + \sin(t) \qquad Eq.48$$

Table 3.2: Estimated and exact value of u(x, t) for example 2.

x	exact	Chebyshev tau	Absolute error
0.1	.91653	.91653	.0
0.2	.99889	.99889	.0
0.3	1.07960	1.07960	.0
0.4	1.15785	1.15785	.0
0.5	1.23285	1.23285	.0
0.6	1.30386	1.30386	.0
0.7	1.39452	1.39452	.0
0.8	1.45871	1.45871	.0
0.9	1.53367	1.53367	.0
1.0	1.64512	1.64512	.0

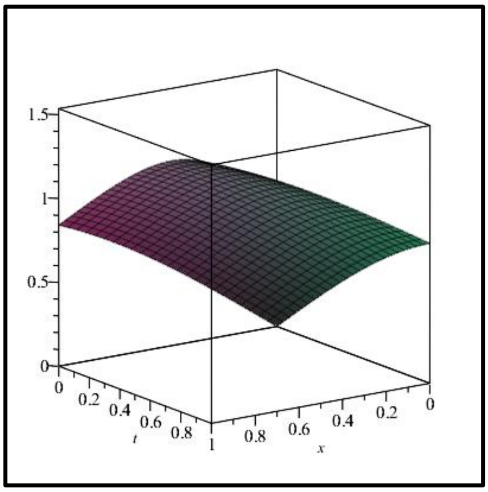

Figure 2: 3D graph of numerical solution for example 2

Based on Table 3.2, the Maple calculation gives almost the same answer as the exact solution. This shows that the Chebyshev-Tau method is efficient and accurate to the exact solution for this example.

4. Conclusion

The background of shifted Chebyshev and Tau method is being explored for the research. Then, a standard Chebyshev-Tau method to solve Klein Gordon also have been studied. Two examples of Klein Gordon equation are solved using the Chebyshev-Tau method and the solutions obtained being compared with the exact value in finding the accuracy of the methods. Maple 2016 was the software used which provides numerical solutions and also graphical output that make the research easier to solve. The numerical solutions obtained from Chebyshev-Tau Method are making a small difference from the exact solutions. So, it can be concluded that Chebyshev-Tau Method can be used to solve linear Klein Gordon equation because the method was accurate with the exact solution. Suggestion for further research for Chebyshev-Tau method are solve the nonlinear Klein Gordon equation and compare Chebyshev-Tau method with other method to study deeper in the accuracy or the methods.

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