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Propagation of Pressure Waves in Inviscid Fluid Contained in Thin Elastic Tube

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Abstract: In this study, the artery is treated as an isotropic, incompressible, thin-walled elastic tube while the blood is assumed as an incompressible inviscid fluid. Inviscid fluid refers to the nonviscous fluid, that is the viscosity of the fluid is equal to zero. Under the assumption of long wave approximation, the reductive perturbation method is adopted to obtain a set of nonlinear differential equations with various orders. By solving these various orders of differential equations, various orders of differential equations are reduced to a nonlinear evolution equation which is called Korteweg-de Vries (KdV) equation. Next, the KdV equation is solved analytically. The graphical outputs have been presented and discussed. It is found that the solution of the KdV equation is in the form of an envelope traveling solitary waves which propagate to the right along the tube. This study is restricted to the propagation of harmonic waves in the inviscid fluid. Therefore, the fluid velocity and fluid pressure maintained the shape of the wave without deformation when the harmonic waves propagated along the tube.

Keywords: Thin-Walled Elastic Tube, Inviscid Fluid, Kdv Equation, Reductive Perturbation Method

1. Introduction

In 1808, Thomas Young is the first person who discovered the pulse waves' speed in human arteries. According to Demiray [1], the researchers considered the artery with different types of tubes while the blood was assumed as different types of fluids. Fu and Il'chev [2] had examined the propagation of the solitary wave in fluid-filled membrane tubes. They have come out with the conclusion that there appear four types of solitary waves that will be existed with speeds close to those given by the linear dispersion relation, no matter the fluid is in the condition of initially stationary or not. The solitary wave solutions are obtained by ignoring the higher-order terms that persist for the full equations system. Il'ichew, Shargatov, and Fu [3] had focused on the nonlinear wave propagation in the fluid-filled hyperelastic membrane tube. The reductive perturbation method has been used and the evolution equation, Korteweg-de Vries (KdV) equation is obtained. In their studies, they found that the wave amplitude is

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a decreasing function of the speed. They also concluded that when the speed is greater, then the solitary waves are spectrally stable. According to Goh and Choy [4], the weakly nonlinear wave is investigated in their study by considering the artery as a thin-walled, prestressed elastic tube with stenosis while the blood is considered as a Newtonian fluid. In their studies, by using the reductive perturbation method, the nonlinear evolution equation was simplified and reduced to the Korteweg-de Vries Burgers (KdVB) equation with variable coefficients. Progressive wave solution for the Korteweg-de Vries Burgers KdVB equation is obtained by using the hyperbolic tangent method. The wave trajectory in terms of its waveform and amplitude is changing when the stenosis in the tube and the viscosity of fluid exists. Therefore, the axial velocity and the amplitude of wave trajectory will increase when the severity of stenosis and level of a viscosity increased.

Moreover, according to Elgarayhi *et al.* [5], they have been studied the propagation of pressure waves in the nonlinear form in fluid-filled elastic tube. During their investigation, the reductive perturbation method is applied and KdV equation is obtained. The effect of the tube with the final inner radius on the basic properties of the soliton waves was investigated. Then, Alasakani, Tantravahi, and Kumar [6] investigate the methods for reducing an input dataset into a developed mathematical model to simulate blood flow through the human artery. The inputs to the parameters are from the physiological information on blood and the anatomical data on arteries. Statistical testing procedures are used to identify significant differences in the independent variables with the values of the dependent variables where the independent variables here are indicated by the model parameters while the values of the dependent variables are computed using the developed mathematical models. According to Wilcox Bunonyo, and Amos [7], lipid concentration effect on blood flow through an inclined arteries channel with the magnetic field is studied. After using the perturbation method, the nonlinear ordinary differential equations are solved analytically. Therefore, the height of stenosis, angle of inclination, length of the stenosis, and rate of the pulse will affect blood flow profile and lipid concentration profile.

As pointed out by Paquerot and Remoissenet [8], the blood viscosity is negligible in some applications, for example, parameters appropriate to biological investigation in main arteries. Hence, for the problems of flow in large blood vessels, blood is treated as an incompressible inviscid fluid as a first approximation. The artery is treated as an isotropic, incompressible, thin-walled elastic tube, the propagation of pressure waves in this medium is investigated by employing the reductive perturbation method. In this present research, after introducing the reductive perturbation method into the equations of tube and fluid, a set of various order of differential equations is obtained. The various order of equations obtained will then be reduced to the Korteweg-de Vries (KdV) equation. The progressive wave solution is implemented to the KdV equation. Next, graphical outputs for the progressive wave solution are presented. Lastly, a discussion about the physical interpretation related to the results obtained is presented. This paper is organized as follows: In Section 2, we introduce the basic equations that govern the model. In Section 3, the long wave approximation is adopted. In Section 4, the solution of the field equations is discussed. In Section 5, a progressive wave solution is implemented. In Section 6, the results and the discussion are given. Lastly, in Section 7, a conclusion is made.

2. Basic Equations

The mathematical model for blood in an elastic tube filled with inviscid fluid is studied. Therefore, this study begins by introducing the equations of tube and fluid by [5].

2.1 Equation of Tube

The artery is set as an isotropic, incompressible, thin-walled elastic tube. Figure 1 illustrated the geometry with a constant radius. The equation of radial motion of the tube is given by [5]

$$\frac{\partial}{\partial z^*}\left\{\frac{\mu}{1+\left[\left(\frac{\partial u^*}{\partial z^*}\right)^2\right]^{\frac{1}{2}}}\frac{\partial \Sigma}{\partial \lambda_1}\frac{\partial u^*}{\partial z^*}\right\}-\frac{\mu}{\lambda_z R_0}\frac{\partial \Sigma}{\partial \lambda_2}+\left(\lambda_\theta+\frac{u^*}{R_0}\right)\frac{P^*}{H}=\frac{\rho_0}{\lambda_z}\frac{\partial^2 u^*}{\partial t^{*2}}, \qquad \text{Eq. 1}$$

where μ denotes the shear modulus of the tube material, Σ denotes the strain energy density function, u^* denotes the radial displacement, z^* denotes the axial coordinate after the static deformation, λ_1 denotes the stretch ratio along the meridional curve, λ_z denotes the axial stretch ratio of the tube, R_0 denotes the radius of the circular cylindrical long tube, λ_2 denotes the stretch ratio along the circumferential curve, λ_θ denotes the stretch ratio in the circumferential direction after finite static deformation, P^* denotes the pressure of the fluid, H denotes the initial tube thickness, ρ_0 denotes the mass density of the tube material, and t^* denotes the parameter of the time.

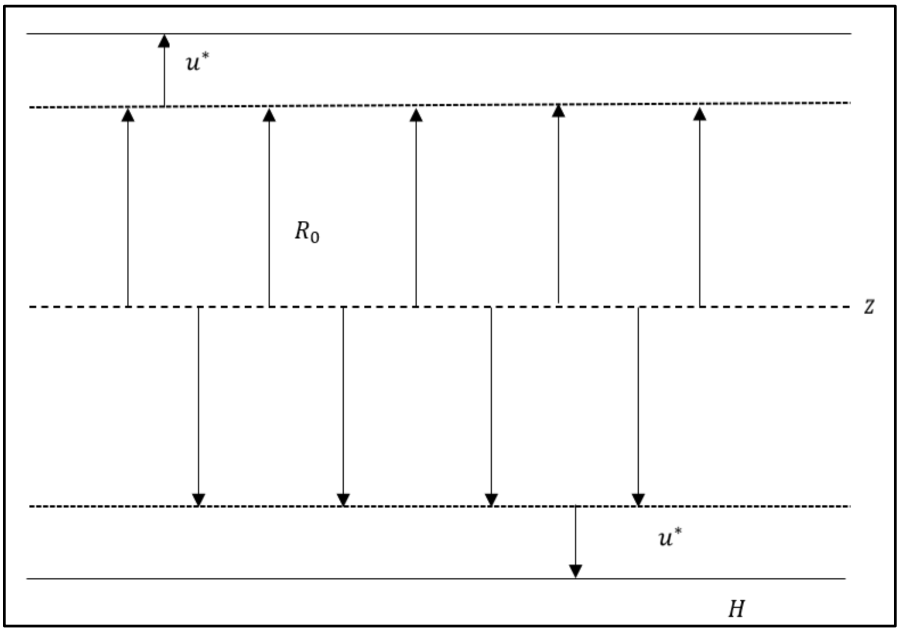

Figure 1: The geometry of the artery with a constant radius

2.2 Equations of Fluid

Generally, blood is a non-Newtonian fluid that is incompressible and has heterogeneous properties as the blood contains fluid plasma and solid components such as platelets, red and white blood cells [9]. Blood behaves like incompressible non-Newtonian fluid due to the deformability of red blood cells, and the level of cell concentration, or called hematocrit ratio. As of simplicity, in this study, blood is treated as an incompressible inviscid fluid. The equations of inviscid fluid in the cylindrical polar coordinates are given by [5]

$$\frac{\partial V_r^*}{\partial r}+\frac{V_r^*}{r}+\frac{\partial V_{z^*}^*}{\partial z^*}=0, \qquad \text{Eq. 2}$$

$$\frac{\partial V_r^*}{\partial t^*}+V_r^*\frac{\partial V_r^*}{\partial r}+V_{z^*}^*\frac{\partial V_r^*}{\partial z^*}+\frac{1}{\rho_f}\frac{\partial \bar{P}}{\partial r}=0, \qquad \text{Eq. 3}$$

$$\frac{\partial V_{z^*}^*}{\partial t^*}+V_r^*\frac{\partial V_{z^*}^*}{\partial r}+V_{z^*}^*\frac{\partial V_{z^*}^*}{\partial z^*}+\frac{1}{\rho_f}\frac{\partial \bar{P}}{\partial z^*}=0, \qquad \text{Eq. 4}$$

where V_r^* is the fluid velocity components in the radial directions, $V_{z^*}^*$ is the fluid velocity components in the axial directions, ρ_f is the fluid mass density, and $\bar{P}$ is the fluid pressure function.

3. Long Wave Approximation

To investigate the propagation of small-but-finite amplitude waves in a thin elastic tube filled with fluid, a long wave approximation is applied and the reductive perturbation is adopted. The following type of stretched coordinates are introduced [5]

$$\xi = \epsilon^{\frac{1}{2}}(z^* - gt^*), \qquad \tau = \epsilon^{\frac{3}{2}} z^*. \tag{Eq. 5}$$

The differential relations are defined as [10]

$$\frac{\partial}{\partial t^*} \to -\epsilon^{\frac{1}{2}} g \frac{\partial}{\partial \xi}, \qquad \frac{\partial}{\partial z^*} \to \epsilon^{\frac{1}{2}}\left(\frac{\partial}{\partial \xi} + \epsilon \frac{\partial}{\partial \tau}\right). \tag{Eq. 6}$$

The asymptotic series are introduced as [5]

$$V_r^* = \sum_{n=1}^{\infty} \epsilon^{\frac{1}{2}+n} V_r^{*(n)}(\xi,\tau,r), \qquad V_{z^*}^* = \sum_{n=1}^{\infty} \epsilon^n V_{z^*}^{*(n)}(\xi,\tau,r),$$
$$\bar{P} = \sum_{n=1}^{\infty} \epsilon^n \bar{P}_{(n)}(\xi,\tau,r), \qquad u^* = \sum_{n=1}^{\infty} \epsilon^n u_n^*(\xi,\tau,r). \tag{Eq. 7}$$

Introducing Eq. 5 – Eq. 7 into Eq. 2 – Eq. 4, the following various order differential equations set are obtained.

$O(\epsilon)$ Equations:

$$\frac{\partial V_r^{*(1)}}{\partial r} + \frac{V_r^{*(1)}}{r} + \frac{\partial V_{z^*}^{*(1)}}{\partial \xi} = 0, \tag{Eq. 8}$$

$$\frac{1}{\rho_f}\frac{\partial \bar{P}_{(1)}}{\partial r} = 0, \tag{Eq. 9}$$

$$-g\frac{\partial V_{z^*}^{*(1)}}{\partial \xi} + \frac{1}{\rho_f}\frac{\partial \bar{P}_{(1)}}{\partial \xi} = 0. \tag{Eq. 10}$$

$O(\epsilon^2)$ Equations:

$$\frac{\partial V_r^{*(2)}}{\partial r} + \frac{V_r^{*(2)}}{r} + \frac{\partial V_{z^*}^{*(2)}}{\partial \xi} + \frac{1}{g\rho_f}\frac{\partial \bar{P}_{(1)}}{\partial \tau} = 0, \tag{Eq. 11}$$

$$\frac{1}{2\rho_f}\frac{\partial^2 \bar{P}_{(1)}}{\partial \xi^2} r + \frac{1}{\rho_f}\frac{\partial \bar{P}_{(2)}}{\partial r} = 0, \tag{Eq. 12}$$

$$-g\frac{\partial V_{z^*}^{*(2)}}{\partial \xi} + \frac{1}{g^2\rho_f^2}\bar{P}_{(1)}\frac{\partial \bar{P}_{(1)}}{\partial \xi} + \frac{1}{\rho_f}\frac{\partial \bar{P}_{(2)}}{\partial \xi} + \frac{1}{\rho_f}\frac{\partial \bar{P}_{(1)}}{\partial \tau} = 0. \tag{Eq. 13}$$

4. Solution of the Field Equations

First, $u_1^* = U(\xi,\tau)$ is assumed by [5]. Then, the boundary conditions are introduced by [5].

$$\bar{P}_{(1)}\Big|_{r=r_f} = \frac{\beta_1 \mu H}{R_0^2} u_1^*, \qquad V_r^{*(1)}\Big|_{r=r_f} = -g\frac{\partial u_1^*}{\partial \xi}. \qquad \text{Eq. 14}$$

From Eq. 8 – Eq. 10, some operations are done to obtain the solution of $\bar{P}_{(1)}(\xi,\tau), U(\xi,\tau), V_{z^*}^{*(1)}$, and $V_r^{*(1)}$. The solutions are obtained as

$$\bar{P}_{(1)}(\xi,\tau) = \frac{\beta_1 \mu H}{R_0^2} U(\xi,\tau), \qquad U(\xi,\tau) = \frac{R_0^2}{\beta_1 \mu H}\bar{P}_{(1)}(\xi,\tau),$$

$$V_{z^*}^{*(1)} = \frac{1}{g\rho_f}\bar{P}_{(1)}(\xi,\tau), \qquad V_r^{*(1)} = -\frac{r}{2g\rho_f}\frac{\partial \bar{P}_{(1)}(\xi,\tau)}{\partial \xi}, \qquad \text{Eq. 15}$$

provided $g^2 = \frac{\beta_1 \mu H r_f}{2\rho_f R_0^2}$, where the unknown function $U(\xi,\tau)$, its governing equation will be acquired later while the function g is corresponding to the phase velocity. From Eq. 12 will get

$$\bar{P}_{(2)} = -\frac{r^2}{4}\frac{\partial^2 \bar{P}_{(1)}}{\partial \xi^2}. \qquad \text{Eq. 16}$$

Then, eliminating the second-order perturbed quantities $V_{z^*}^{*(2)}$ and $\bar{P}_{(2)}$, the acquired Korteweg-de Vries (KdV) equation is shown as follow

$$\frac{\partial \bar{P}_{(1)}}{\partial \tau} + A\bar{P}_{(1)}\frac{\partial \bar{P}_{(1)}}{\partial \xi} + B\,\frac{\partial^3 \bar{P}_{(1)}}{\partial \xi^3} = 0, \qquad \text{Eq. 17}$$

where the coefficients are defined by

$$A = \frac{R_0^2}{\beta_1 \mu H}\left(\frac{\beta_1 \mu H}{2g^2\rho_f R_0^2} + \frac{2g^2\rho_f\beta_2 R_0}{\beta_1^2 \mu H r_f} + \frac{1}{r_f}\right),$$

$$B = \frac{g^2\rho_f R_0^2 r_f}{4\beta_1 \mu H} + \frac{g^4\rho_0\rho_f R_0^4}{\lambda_\theta\lambda_z\beta_1^2\mu^2 H r_f} - \frac{r_f^2}{16} - \frac{\alpha_0 g^2 R_0^4 \rho_f}{\beta_1^2 \mu H r_f}. \qquad \text{Eq. 18}$$

5. Progressive Wave Solution

Eq. 17 has the progressive wave solution as follow [5]:

$$\eta = (\xi - v\tau) \equiv \bar{P}_{(1)}(\eta)\,. \qquad \text{Eq. 19}$$

Now, it is required that in the case that $\eta \to \pm\infty$, one can get

$\bar{P}_{(1)} = 0, \frac{d\bar{P}_{(1)}}{d\eta} = 0$, and $\frac{d^2\bar{P}_{(1)}}{d\eta^2} = 0$ by [5]. The evolution equation $\bar{P}_{(1)}$ has the following type of progressive wave

$$\bar{P}_{(1)} = \bar{P}_{(0)} sech^2\left[\frac{\eta}{\Delta}\right], \qquad \text{Eq. 20}$$

where $\Delta = 2\sqrt{\frac{B}{v}}$, and $\bar{P}_{(0)} = \frac{3v}{A}$.

6. Results and Discussion

The propagation of pressure waves in inviscid fluid contained in thin elastic tube has been studied. The graphical outputs and the discussion are presented in this section. $\beta_1 = 296.105$, $\beta_2 = 991.496$, $\rho_f = \frac{1.05gm}{cm^3}$, $R_0 = 0.38cm$, $\rho_0 = \frac{1.03gm}{cm^3}$, $\lambda_z = \lambda_\theta = 1.6$, $\alpha_0 = 78.692$, $H = 2 \times$

$10^{-2}cm$, $\mu = 0.4$, $v = 8cm/s$, and $r_f = 0.75$ are used in order to obtain the graphical outputs [5]. Figures 2 - 5 illustrate the space for $-0.2 < \tau < 1.5$ and traveling wave profile, $0 < \xi < 10$.

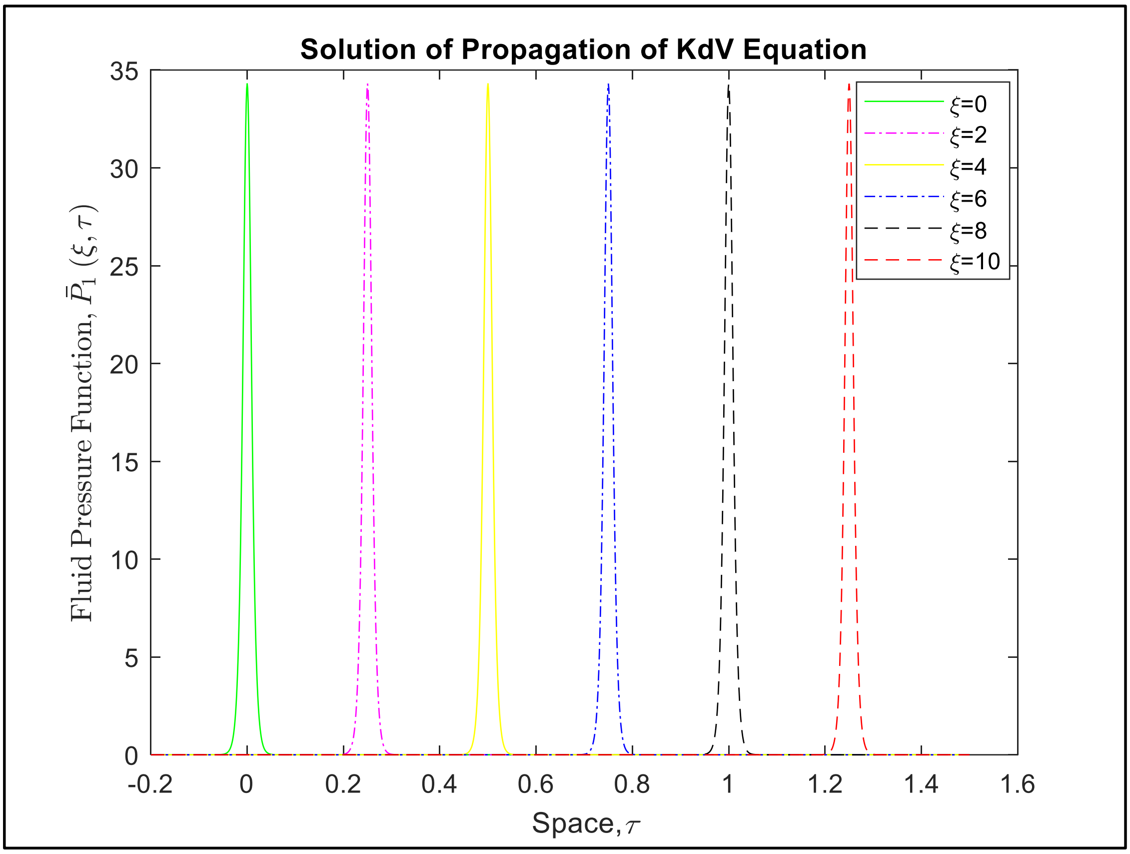

Figure 2: The fluid pressure function, $\bar{P}_{(1)}(\xi,\tau)$ based on Eq. 20 versus spaces, τ for different wave profiles, ξ.

Figure 2 illustrates the variation of the analytical solution of the KdV based on Eq. 20 with space. As seen from the figure due to the absence of viscous effect in fluid, the amplitude of the fluid pressure function unchanged with increasing traveling wave profile.

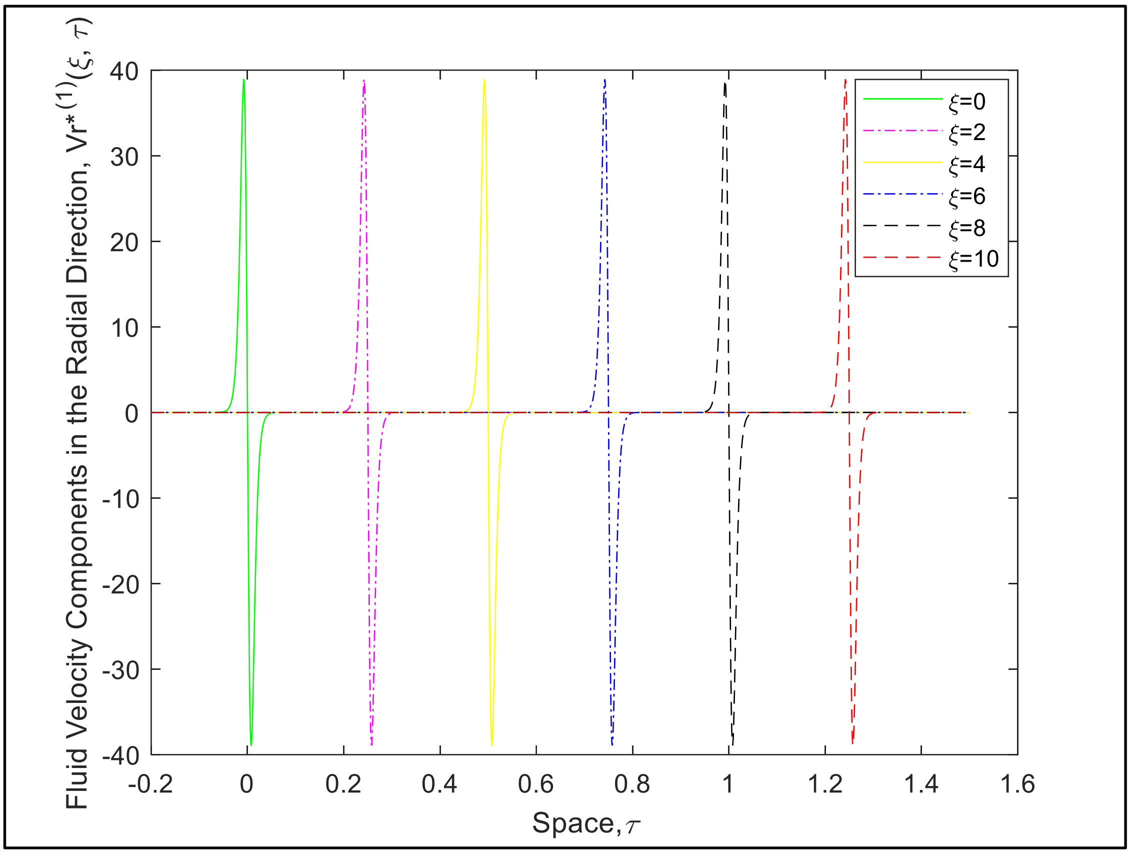

Figure 3: The fluid velocity components in the radial direction, $V_r^{*(1)}(\xi,\tau)$ based on Eq. 15 versus spaces, τ for different wave profiles, ξ.

Figure 3 reveals the solution of fluid velocity components in the radial direction based on Eq. 15 with space. It shows the variation of kink wave soliton solution with increasing wave profile. Again, due to the without viscous effect in the fluid, the solitary waves propagated to the right with a permanent kink shape wave structure.

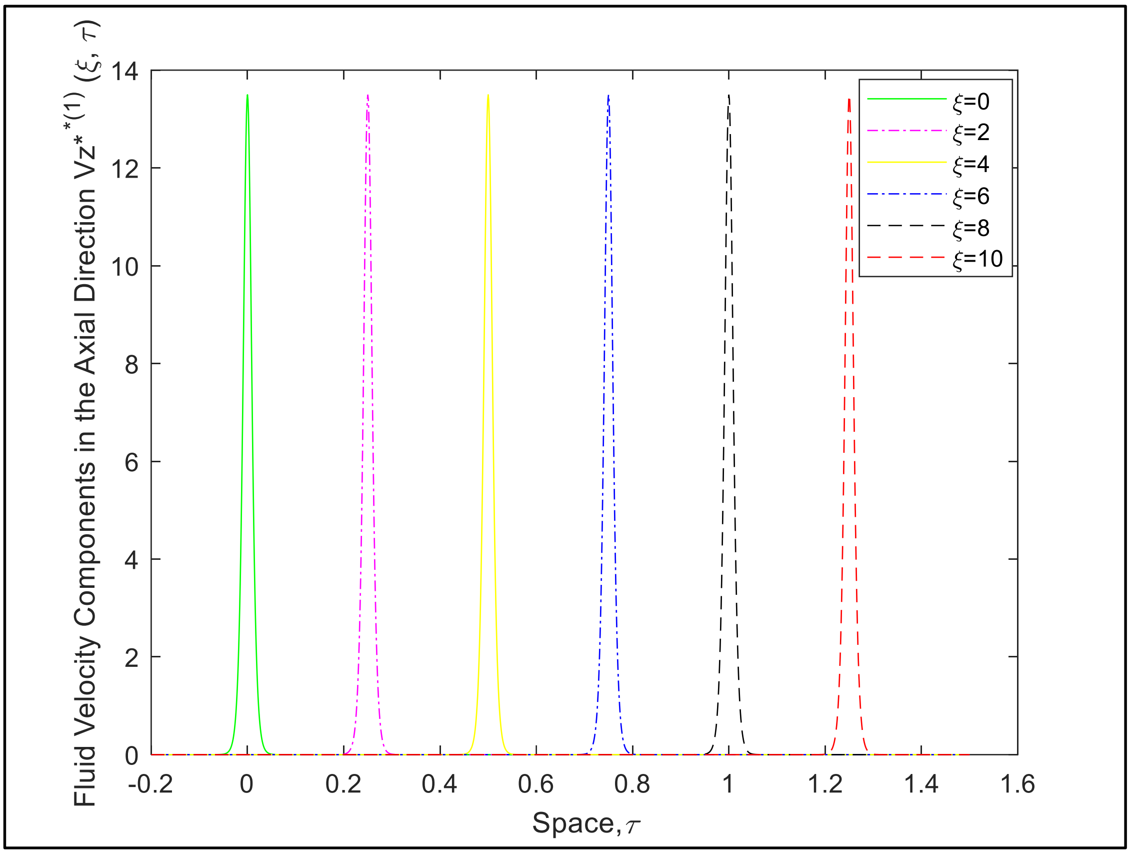

Figure 4: The fluid velocity components in the axial direction, $V_{z^*}^{*(1)}(\xi,\tau)$ based on Eq. 15 versus spaces, τ for different wave profiles, ξ.

Figure 4 displays the results of the fluid velocity components in the axial direction based on Eq. 15 at a particular value of the traveling wave profile. The symmetrical bell-shaped waves structure does not change when these solitary waves travel along the tube. This is because the resistance to the flow of an inviscid fluid does not occur.

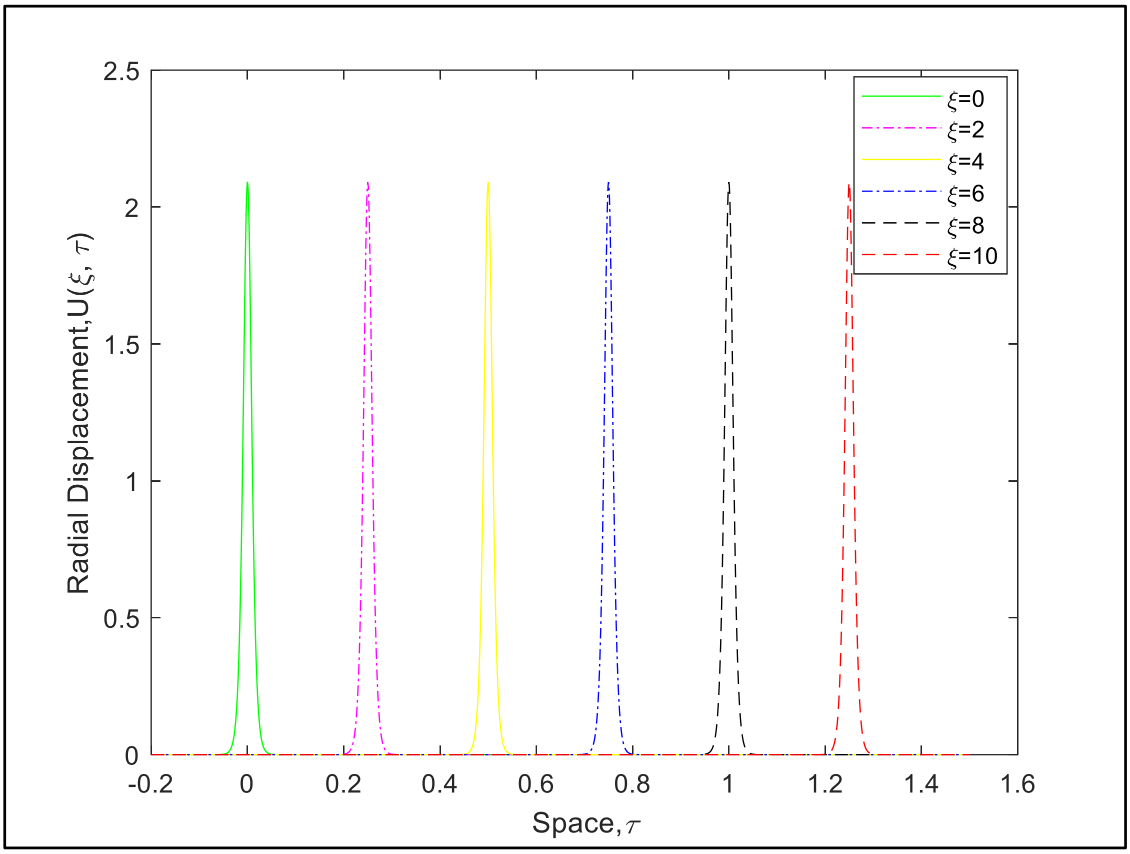

Figure 5: The radial displacement, $U(\xi,\tau)$ based on Eq. 15 versus spaces, τ for different wave profiles, ξ.

Figure 5 presents the radial displacement based on Eq. 15 through inviscid fluid along the thin elastic tube. This figure shows that the amplitude of waves is always 2.0909 with the increasing traveling wave profile. It can be concluded that the amplitude of the wave does not change when the viscosity is not considered.

In reality, blood is known to be an incompressible non-Newtonian fluid. The viscosity changes regarding the temperature, pressure, and density. However, in some applications, when blood flows in the large blood vessels, as first approximation, the effect of viscosity is neglected. Therefore, from Figures 2 – 5, it is realized that the solution of KdV equation preserves its wave structure without reduction of wave amplitude. This is due to the resistance of fluid flow does not exist in the inviscid fluid. Besides, the occurrence of a stable structure for the amplitude of the solitary wave is due to the exhibits of balance nonlinearity and dispersion effects.

7. Conclusion

As a summary, the reductive perturbation method is used in this study in order to investigate the wave propagation in the thin elastic tube filled with inviscid fluid. The governing equation for the corresponding mathematical model is Korteweg-de Vries (KdV) equation. Next, the KdV equation has been solved analytically. It can be concluded that the fluid pressure function, radial displacement, and fluid velocity in the axial direction admit solitary wave solutions. The waves propagate to the right by retaining their bell-shaped wave. The fluid velocity in the radial direction shows the variation of the kink wave structure. These kink waves travel to the right without deformation. Hence, one can conclude that nonlinear wave propagation in inviscid fluid does not influence the solitary wave solution.

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References

[1] Demiray, H. (2007). Waves in fluid-filled elastic tubes with a stenosis: Variable coefficient KdV equations. *Journal of Computational and Applied Mathematics*, *202*(2), 328-338.

[2] Fu, Y., & Il'ichev, A. (2010). Solitary waves in fluid-filled elastic tubes: existence, persistence, and the role of axial displacement. *IMA Journal of Applied Mathematics*, *75*(2), 257-268.

[3] Il'ichev, A. T., Shargatov, V. A., & Fu, Y. B. (2020). Characterization and dynamical stability of fully nonlinear strain solitary waves in a fluid-filled hyperelastic membrane tube. *Acta Mechanica*, *231*(10), 4095-4110.

[4] Goh, T. H., and Choy, Y. Y. (2020). Mathematical Formulation for Nonlinear Wave Propagation in a Stenosed Elastic Tube. *Communications in Computational and Applied Mathematics*, *2*(1). 19-24.

[5] Elgarayhi, A., El-Shewy, E. K., Mahmoud, A. A., & Elhakem, A. A. (2013). *Propagation of nonlinear pressure waves in blood*. Egypt: Mansoura University.

[6] Alasakani, K., Tantravahi, R. S., & Kumar Ptv, P. (2021). On Refining the Input Data set to Mathematical Models Simulating Arterial blood flow in Humans. WSEAS TRANSACTIONS ON FLUID MECHANICS, 16, 63–78. https://doi.org/10.37394/232013.2021.16.7.

[7] Wilcox Bunonyo, K., & Amos, E. (2020). Lipid Concentration Effect on Blood Flow Through an Inclined Arterial Channel with Magnetic Field. Mathematical Modelling and Applications, 5(3), 129. https://doi.org/10.11648/j.mma.20200503.11.

[8] Paquerot, J. F., & Remoissenet, M. (1994). Dynamics of nonlinear blood pressure waves in large arteries. Physics Letters A, 194(1–2), 77–82. https://doi.org/10.1016/0375-9601(94)00729-9.

[9] Smithee, I., & Gent, S. P. (2018). Computational Fluid Dynamics Modeling of Blood As a Heterogeneous Fluid. *2018 Design of Medical Devices Conference*. Published. https://doi.org/10.1115/dmd2018-6873.

[10] Gaik, T. K., & Demiray, H. (2008). Forced Korteweg-de Vries–Burgers equation in an elastic tube filled with a variable viscosity fluid. *Chaos, Solitons & Fractals*, *38*(4), 1134–1145. https://doi.org/10.1016/j.chaos.2007.02.005.