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The Effectiveness of Differential Transform Method (DTM) on Solving the Duffing Equation

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Abstract: In this paper, a semi-analytical solution called differential transform method (DTM) is applied to solve the Duffing equation. We solved examples of Duffing equation with the homogeneous, linear and trigonometric function. The solution is obtained semi analytically by using the DTM with aid of Excel and Maple 15. Maple 15 also used to compare the DTM solution and the exact solution by displaying the numerical results and graphical outputs. The accuracy of the DTM is compared to the exact solution or other methods done by previous researcher. By the end of the research, it showed that the DTM is able to solve the Duffing equation and obtain result close to the exact solution. However, convergence speed towards the exact solution only to small parameter.

Keywords: Differential Transform Method, Duffing Equation

1. Introduction

The differential transform method (DTM) is a semi-analytical method to solve differential equations and developed in polynomial form based on expansions of Taylor series. DTM was introduced by Zhau in 1986 [1]. He found out that Taylor series is too complicated to solve higher order differential equation problems. Therefore, he introduced new form of Taylor series which known as DTM to solve mathematical problem in electric circuit analysis. DTM has a lot of benefits. One of them was it can be applied directly to linear and nonlinear ordinary differential equations and got a fast convergence rate and a small calculation error. DTM also provide a series solution that will efficiently converge to the approximate solution.

Duffing equation is a second order nonlinear ordinary differential equation in the oscillator system. George Duffing invented the Duffing equation. The Duffing equation was well-known among engineers since he was one of them. Because the existence of y^3 in the equation, it is called a nonlinear equation.

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The general form of the Duffing equation has been written in the form

$$y''(x) + py'(x) + p_1 y(x) + p_2 y^3(x) = f(x) \quad \text{Eq. 1}$$

with the initial conditions

$$y(0) = \alpha, \qquad y'(0) = \beta, \quad \text{Eq. 2}$$

where p, p_1, p_2, α and β are real constants.

The aim of this paper is to apply DTM to obtain the approximate solutions of Duffing equations. We demonstrate the accuracy of the DTM through some test examples. Numerical comparison will be made against the exact solution or other methods.

2. Methodology

We will illustrate the basic ideas of DTM. Consider the following nonlinear Duffing equation in form Eq. 1 and Eq. 2. It is commonly known if a function $f(x)$ is definitely continuously differentiable, then the differential transform of the $f(x)$ function for the kth derivative is defined as follows:

$$f(x) = \sum_{k=0}^{\infty} \frac{(x-x_0)^k}{k!} \left. \frac{d^k f(x)}{dx^k} \right|_{x=x_0}, \quad \text{Eq. 3}$$

The fundamental operations performed by differential transform can readily be obtained and are listed in Table 1.

Table 1 : Operational properties of differential transformation method [2].

Original function	Transformed function
$f(x) = u(x) \pm v(x)$	$F(k) = U(k) \pm V(k)$
$f(x) = \alpha u(x)$	$F(k) = \alpha U(k)$
$f(x) = u(x)v(x)$	$F(k) = \sum_{i=0}^{k} V(i)U(k-i)$
$f(x) = \frac{du(x)}{dx}$	$F(k) = (k+1)U(k+1)$
$f(x) = \frac{d^m u(x)}{dx^m}$	$F(k) = (k+1)(k+2)...(k+m)U(k+m)$
$f(x) = u^m$	$F(k) = \partial(k-m) = \begin{cases} 1, if\ k = m \\ 0, if\ k \neq m \end{cases}$
$f(x) = u^3(x)$	$F(k) = \sum_{i=0}^{k}\sum_{j=0}^{k} U(i)U(j)U(k-i-j)$
$f(x) = (1+u)^m$	$F(k) = \frac{m(m-1)...(m-k+1)}{k!}$
$f(x) = \exp(\lambda x)$	$F(k) = \frac{\lambda^k}{k!}$
$f(x) = \sin(\omega x + \alpha)$	$F(k) = \frac{\omega^k}{k!}\sin\left(\frac{\pi k}{2!} + \alpha\right)$
$f(x) = \cos(\omega x + \alpha)$	$F(k) = \frac{\omega^k}{k!}\cos\left(\frac{\pi k}{2!} + \alpha\right)$

2.1 Duffing Equation with Homogeneous Function

Consider the Duffing equation [3] as shown below:

$$y''(x) - 3y'(x) + 2y(x) - 2y^3(x) = 0 \qquad \text{Eq. 4}$$

with the initial conditions

$$y(0) = \frac{1}{2}, y'(0) = \frac{1}{4} \qquad \text{Eq. 5}$$

and given the exact solution

$$y(x) = \frac{1}{1+e^{-x}} \qquad \text{Eq. 6}$$

By applying DTM to Eq. 4 from Table 1, we will obtain as follows

$$(k+1)(k+2)Y(k+2) - 3[(k+1)Y(k+1)] + 2Y(k) - 2\left[\sum_{i=0}^{k}\sum_{j=0}^{k-i} Y(i)Y(j)Y(k-i-j)\right] = 0 \qquad \text{Eq. 7}$$

By applying Table 1, the initial conditions in Eq. 5 can be transformed as

$$Y(0) = \frac{1}{2}, Y(1) = \frac{1}{4} \qquad \text{Eq. 8}$$

Substituting $k = 0$ and the initial conditions in Eq. 8 into Eq. 7 to obtain the second term

$$(0+1)(0+2)Y(0+2) - 3[(0+1)Y(0+1)] + 2Y(0) - 2\left[\sum_{i=0}^{0}\sum_{j=0}^{0-i} Y(i)Y(j)Y(0-i-j)\right] = 0$$

$$2Y(2) = 0$$

$$Y(2) = 0 \qquad \text{Eq. 9}$$

Continue substituting $k = 1$ to $k = 5$ and the initial conditions in Eq. 8 into Eq. 9 to obtain the third term and so on.

Table 2 : Numerical solution of first term until seventh term by using DTM.

k	$Y(k)$
0	$\frac{1}{2}$
1	$\frac{1}{4}$
2	0
3	$-\frac{1}{48}$
4	0
5	$\frac{1}{480}$
6	0
7	$-\frac{17}{80640}$

By using the solutions from Table 2, combine all the terms that were obtained and do the series solution of Taylor series up to seventh term.

$$Y(x)=\sum_{k=0}^{\infty}Y(k)x^k$$
$$=Y(0)x^0+Y(1)x^1+Y(2)x^2+Y(3)x^3+Y(4)x^4+Y(5)x^5+Y(6)x^6+Y(7)x^7+\ldots \quad \text{Eq. 10}$$

we obtain the Taylor series such as

$$Y(x)=\frac{1}{2}+\frac{1}{4}x-\frac{1}{48}x^3+\frac{1}{480}x^5-\frac{17}{80640}x^7+\ldots \quad \text{Eq. 11}$$

After that, we compute numerical solution of DTM using Excel. The result obtained as in Table 5. Figure 1 illustrate the graphical output of the results.

2.2 Duffing Equation with Linear Function

Consider the Duffing equation [4] as shown below:

$$y''(x)+2y'(x)+y(x)+8y^3(x)=1-3x \quad \text{Eq. 12}$$

with the initial conditions

$$y(0)=\frac{1}{2},\ y'(0)=-\frac{1}{2} \quad \text{Eq. 13}$$

and given the exact solution

$$y(x)=\frac{1}{2}e^{-x} \quad \text{Eq. 14}$$

By applying DTM to Eq. 12 from Table 1, we will obtain as follows

$$(k+1)(k+2)Y(k+2)+2[(k+1)Y(k+1)]+Y(k)$$
$$+8\left[\sum_{i=0}^{k}\sum_{j=0}^{k-i}Y(i)Y(j)Y(k-i-j)\right]=\delta(k)-3\sum_{i=0}^{k}\delta(k-1) \quad \text{Eq. 15}$$

By applying Table 1, the initial conditions in Eq. 13 can be transformed as

$$Y(0)=\frac{1}{2},Y(1)=-\frac{1}{2} \quad \text{Eq. 16}$$

Substituting $k=0$ and the initial conditions in Eq. 16 into Eq. 15 to obtain the second term

$$(0+1)(0+2)Y(0+2)+2[(0+1)Y(0+1)]+Y(0)$$
$$+8\left[\sum_{i=0}^{0}\sum_{j=0}^{0-i}Y(i)Y(j)Y(0-i-j)\right]=\delta(0)-3\sum_{i=0}^{0}\delta(0-1)$$
$$2Y(2)+\frac{1}{2}=0$$
$$Y(2)=-\frac{1}{4} \quad \text{Eq. 17}$$

Continue substituting $k=1$ to $k=5$ and the initial condition in Eq. 16 into Eq. 17 to obtain the third term and so on.

Table 3 : Numerical solution of first term until seventh term by using DTM.

k	$Y(k)$
0	$\frac{1}{2}$
1	$-\frac{1}{2}$
2	$-\frac{1}{4}$

3	$\frac{11}{12}$
4	$-\frac{9}{16}$
5	$-\frac{47}{240}$
6	$\frac{847}{1440}$
7	$-\frac{187}{672}$

By using the solution from Table 3, combine all the terms that were obtained and do the series solution of Taylor series in Eq. 10.

we obtain the Taylor series such as

$$Y(x)=\frac{1}{2}-\frac{1}{2}x-\frac{1}{4}x^2+\frac{11}{12}x^3-\frac{9}{16}x^4-\frac{47}{240}x^5+\frac{847}{1440}x^6-\frac{187}{672}x^7+... \quad \text{Eq. 18}$$

After that, we compute numerical solution of DTM using Excel. The result obtained as in Table 6. Figure 2 illustrate the graphical output of the results. Then, we compare DTM solution with exact solution and Daftardar and Jafari method (DJM) in Table 7 with aid of Maple 15. The result obtained in Figure 3.

2.3 Duffing Equation with Trigonometric Function

Consider the Duffing equation [5] as shown below:

$$y''(x)+3y(x)-2y^3(x)=\cos(x)\ \sin(x) \quad \text{Eq. 19}$$

with the initial conditions

$$y(0)=0, y'(0)=1 \quad \text{Eq. 20}$$

and given the exact solution

$$y(x)=\sin(x) \quad \text{Eq. 21}$$

By applying DTM to Eq. 19 from Table 1, we will obtain as follows

$$(k+1)(k+2)Y(k+2)+3Y(k)+8\left[\sum_{i=0}^{k}\sum_{j=0}^{k-i}Y(i)Y(j)Y(k-i-j)\right]$$

$$=\left[\frac{1^k}{k!}\cos\left(\frac{\pi k}{2!}\right)\right]\left[\frac{2^k}{k!}\sin\left(\frac{\pi k}{2!}\right)\right] \quad \text{Eq. 22}$$

By applying Table 1, the initial condition in Eq. 20 can be transformed as

$$Y(0)=0, Y(1)=1 \quad \text{Eq. 23}$$

Substituting $k=0$ and the initial condition in Eq. 23 into Eq. 22 to obtain the second term

$$(0+1)(0+2)Y(0+2)+3Y(0)$$

$$-2\left[\sum_{i=0}^{0}\sum_{j=0}^{0-i}Y(i)Y(j)Y(0-i-j)\right]=\left[\frac{1^0}{0!}\cos\left(\frac{0}{2!}\right)\right]\left[\frac{2^0}{0!}\sin\left(\frac{0}{2!}\right)\right]$$

$$2Y(2)=0$$

$$Y(2)=0 \quad \text{Eq. 24}$$

Continue substituting $k=1$ to $k=5$ and the initial conditions in Eq. 23 into Eq. 24 to obtain the third term and so on.

Table 4: Numerical solution of first term until seventh term by using DTM.

k	$Y(k)$
0	0
1	1
2	0
3	$-\frac{1}{2}$
4	0
5	$\frac{3}{40}$
6	0
7	$-\frac{3}{560}$

By using the solution from Table 4, combine all the terms that were obtained and do the series solution of Taylor series in Eq. 10.

we obtain the Taylor series such as

$$Y(x) = x - \frac{1}{2}x^3 + \frac{3}{40}x^5 - \frac{3}{560}x^7 + ... \quad \text{Eq. 25}$$

After that, we compute numerical solution of DTM using Excel. The result obtained as in Table 8. Figure 4 illustrate the graphical output of the results. Then, we compare DTM solution with exact solution and improved Taylor Matrix method (ITMM) in Table 9 with aid of Maple 15. The result obtained in Figure 5.

3. Results and Discussion

The results obtained demonstrate the effectiveness of DTM for solving the nonlinear Duffing equations. The numerical solution obtained as follows:

Table 5 : Numerical solution of DTM, exact solution of 2.1 and absolute error of DTM.

x	DTM solution	Exact solution [3]	Absolute error
0	0.5000000000	0.5000000000	0
0.1	0.5249791875	0.5249791875	0
0.2	0.5498339973	0.5498339973	0
0.3	0.5744425164	0.5744425168	4.00×10^{-10}
0.4	0.5986876546	0.5986876601	5.50×10^{-9}
0.5	0.6224592905	0.6224593312	4.07×10^{-8}
0.6	0.6456560986	0.6456563062	2.08×10^{-7}
0.7	0.6681869511	0.6681877722	8.21×10^{-7}
0.8	0.6899717892	0.6899744811	2.69×10^{-6}
0.9	0.7109418561	0.7109495026	7.65×10^{-6}
1.0	0.7310391865	0.7310585786	1.94×10^{-5}
1.1	0.7502152466	0.7502601056	4.49×10^{-5}
1.2	0.7684286171	0.7685247853	9.62×10^{-5}
1.3	0.7856416141	0.7858349830	1.93×10^{-4}

1.4	0.8018157411	0.8021838856	3.68×10^{-4}
1.5	0.8169058664	0.8175744762	6.69×10^{-4}

Table 6 : Numerical solution of DTM, exact solution of 2.2 and absolute error of DTM.

x	DTM solution	Exact solution [4]	Absolute error
0	0.5000000000	0.5000000000	0
0.1	0.4483590187	0.4524187090	4.06×10^{-3}
0.2	0.3964047492	0.4093653765	1.30×10^{-2}
0.3	0.3475858103	0.3704091103	2.28×10^{-2}
0.4	0.3042146540	0.3351600230	3.09×10^{-2}
0.5	0.2678238157	0.3032653299	3.54×10^{-2}
0.6	0.2395249143	0.2744058181	3.49×10^{-2}
0.7	0.2202301517	0.2482926519	2.81×10^{-2}
0.8	0.2105960635	0.2246622821	1.41×10^{-2}
0.9	0.2105492683	0.2032828299	7.27×10^{-3}
1.0	0.2182539683	0.1839397206	3.43×10^{-2}
1.1	0.2283809488	0.1664355419	6.19×10^{-2}
1.2	0.2295378286	0.1505971060	7.89×10^{-2}
1.3	0.2007203098	0.1362658965	6.45×10^{-2}
1.4	0.1066441778	0.1232984820	1.67×10^{-2}
1.5	0.1081821987	0.1115650801	2.20×10^{-1}

Table 7 : Numerical solution of DJM, exact solution of 2.2 and absolute error of DJM.

x	DJM solution [4]	Exact solution [4]	Absolute error
0	0.5000000000	0.5000000000	0
0.1	0.4524187500	0.4524187090	4.10×10^{-8}
0.2	0.4093666667	0.4093653765	1.29×10^{-6}
0.3	0.3704187500	0.3704091103	9.64×10^{-6}
0.4	0.3352000000	0.3351600230	4.00×10^{-5}
0.5	0.3033854167	0.3032653299	1.20×10^{-4}
0.6	0.2747000000	0.2744058181	2.94×10^{-4}
0.7	0.2489187500	0.2482926519	6.26×10^{-4}
0.8	0.2258666667	0.2246622821	1.20×10^{-3}
0.9	0.2054187500	0.2032828299	2.14×10^{-3}
1.0	0.1875000000	0.1839397206	3.56×10^{-3}
1.1	0.1720854167	0.1664355419	5.65×10^{-3}
1.2	0.1592000000	0.1505971060	8.60×10^{-3}
1.3	0.1489187500	0.1362658965	1.27×10^{-2}
1.4	0.1413666667	0.1232984820	1.81×10^{-2}
1.5	0.1367187500	0.1115650801	2.52×10^{-2}

Table 8 : Numerical solution of DTM, exact solution of 2.3 and absolute error of DTM.

x	DTM solution	Exact solution [5]	Absolute error
0	0.0000000000	0.0000000000	0
0.1	0.0995007495	0.0998334166	3.33×10^{-4}
0.2	0.1960239314	0.1986693308	2.65×10^{-3}
0.3	0.2866810784	0.2955202067	8.84×10^{-3}
0.4	0.3687592229	0.3894183423	2.07×10^{-2}
0.5	0.4398018973	0.4794255386	3.96×10^{-2}
0.6	0.4976820343	0.5646424734	6.70×10^{-2}
0.7	0.5406640663	0.6442176872	1.04×10^{-1}
0.8	0.5674525257	0.7173560909	1.50×10^{-1}
0.9	0.5772244452	0.7833269096	2.06×10^{-1}
1.0	0.5696428571	0.8414709848	2.72×10^{-1}

Table 9 : Numerical solution of ITMM, exact solution of 2.3 and absolute error of ITMM [5].

x	ITMM solution	Exact solution	Absolute error
0	0.0000000000	0.0000000000	0
0.1	0.0998334166	0.0998334166	4.33×10^{-14}
0.2	0.1986693308	0.1986693308	1.03×10^{-13}
0.3	0.2955202067	0.2955202067	1.66×10^{-13}
0.4	0.3894183423	0.3894183423	2.21×10^{-13}
0.5	0.4794255386	0.4794255386	2.71×10^{-13}
0.6	0.5646424734	0.5646424734	3.15×10^{-13}
0.7	0.6442176872	0.6442176872	2.29×10^{-13}
0.8	0.7173560909	0.7173560909	3.85×10^{-13}
0.9	0.7833269096	0.7833269096	9.73×10^{-13}
1.0	0.8414709848	0.8414709848	1.57×10^{-11}

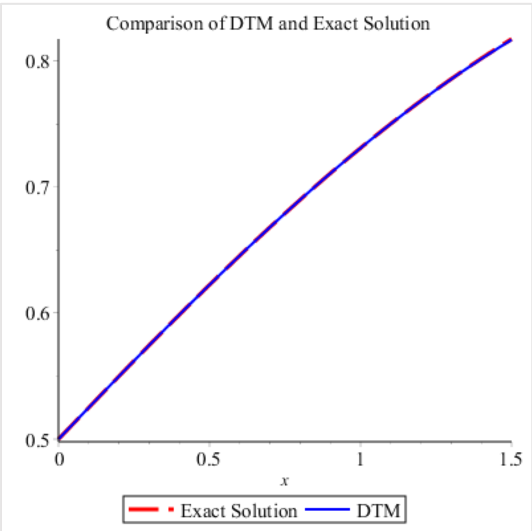

Figure 1: Graph comparison of DTM solution with exact solution of 2.1.

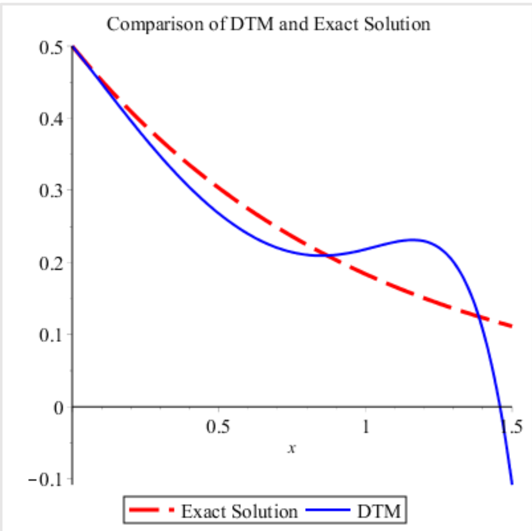

Figure 2: Graph comparison of DTM solution with exact solution of 2.2.

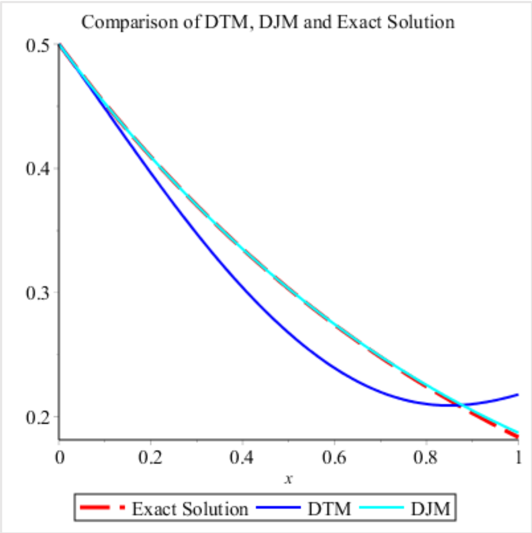

Figure 3: Graph comparison of DTM solution, DJM solution with exact solution of 2.2.

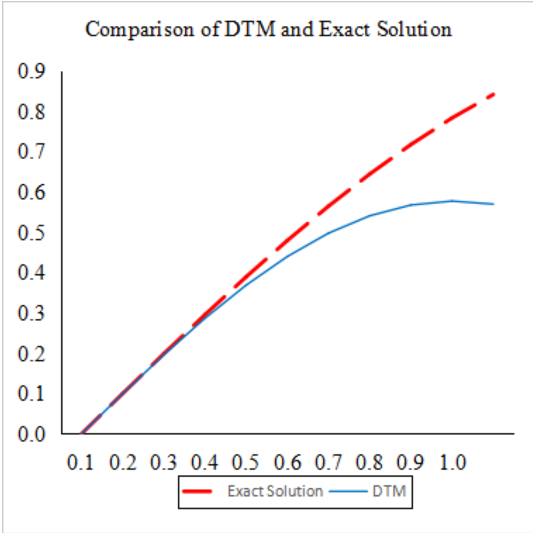

Figure 4: Graph comparison of DTM solution with exact solution of 2.3.

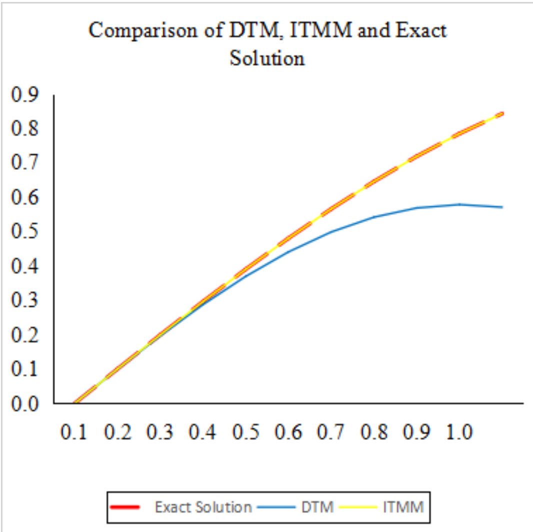

Figure 5: Graph comparison of DTM solution, ITMM solution with exact solution of 2.3.

4. Conclusion

The higher term of DTM can improve the approximation towards the exact solution. However, the higher term of DTM is required more computational work. For solving Duffing equation, only a few terms is recommended to obtain the approximate value. The objectives for this research are to explore method of DTM for solving nonlinear Duffing equation, finding the analytical solution of Duffing equation by using DTM and compare the accuracy of DTM with exact solutions or other methods. The problem has been solved using DTM but it is only reliable for homogeneous function (2.1) and restricted to a small region of x for non-homogeneous function (2.2 and 2.3). DTM is the simplest method to apply because of less computational work to get approximation and can solve any nonlinear differential problem. Thus, some modifications to DTM must be made due to overcome these limitations. However, these are not in the scope of our study so then we can conclude that all the objectives of this study are achieved. There are some recommendations that can be made for further study and research on DTM which are improve the algorithm of differential transform method to obtain simple short term to obtain exact solution and using modified differential transform method with Adomain polynomials or Laplace transform or Padé approximation [6] to improve the solution of close to exact solution.

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