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Solitary Waves in Stenosed Thin Elastic Tube with Variable Viscosity Newtonian Fluid

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Abstract: For this study, the artery is considered as a prestressed thin-walled elastic stenosed tube, moreover the blood is treated as an incompressible Newtonian fluid with variable viscosity. Here, the solitary wave propagation in this composite medium has been investigated by using the reductive perturbation method. A set of various orders of nonlinear differential equations are obtained by introducing the reductive perturbation method into the dimensionless equations (tube and fluid). Then, the various orders of differential equations are solved to get the forced Korteweg-de Vries-Burgers (FKdVB) equation with variable coefficient. The evolution equation is solved analytically. The result revealed that when the blood flows in a stenosed tube, the wave amplitude decreases over time corresponding to the viscous effect of fluid and the stenosis. Conversely, when blood flows in a tube without stenosis, the wave structure is an increasing shock wave profile propagates to the right. In addition, by discarding the stenotic effect, the solution of fluid pressure shows an increasing shock wave profile propagates to the right when the time increases. The fluid pressure function reached minimum value in the center of stenosis due to the existence of stenosis. The wave speed variation is presented when different value of stenotic effects is under consideration.

Keywords: Stenosed Tube, Newtonian Fluid With Variable Viscosity, Reductive Perturbation Method, Fkdvb Equation With Variable Coefficient

1. Introduction

A circulatory system is known as the cardiovascular system that includes the heart and blood vessels and contains about 11 pints (5 liters) of blood [1]. In the arteries system, the total volume of blood held is around 10% to 15% [2]. The blood flow in the human body was found by William Harvey [3]. Blood exhibits Newtonian behaviour in most arteries, and the viscosity can be considered constant [4]. In addition, Faivre recorded that the human body also consists of pressure in the blood [5]. In 1808, Thomas Young was the first who derive the speed of pulse waves in an elastic tube containing an incompressible liquid [6].

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Since Thomas Young discovered this pulse wave speed, the nonlinear propagating of waves in a tube filled fluid became a popular research target globally. The related studies have been done by Malflient and Ndayirinde [7], Bakirtas and Demiray [8], Tay [9], and Il'íchev, Shargatov, and Fu [10]. From these studies, researchers focused on the solitary waves in a inviscid fluid-filled thin elastic tube by using tanh method or reductive perturbation method to achieve the governing equation.

Some studies related to the mathematical model for arterial blood flow have been done. Bakirtas and Antar [11], Tay, Ong and Mohamad [12], Demiray [13], Nikolova [14], Goh and Choy [15] and Yang, Song and Yang [16] obtained the KdV equation through the reductive perturbation method. Bakirtas and Antar [11] treated the artery as an elastic, stenosed, and thin-walled long tube, while the blood considered to be an incompressible inviscid fluid. Yang, Song and Yang [16] considered the artery as an elastic deformable tube and blood is treated as inviscid fluid. However, the other researchers studied the propagation of wave in a bumped prestressed thin elastic wall with Newtonian fluid. Furthermore, Gao and Zhang [17] reviewed a thin elastic tube with viscous fluid by using a new model of the multiscale analysis and perturbation method to obtain a Boussinesq equation. Besides, Ali, Hussain, Anwar and Inc [18] study the blood flow in an artery with stenosis by using numerical method, named finite difference method. Kumar and Choy [19] explored the solitary wave modulation in an elastic tube with variable radius with inviscid fluid by implement the reductive perturbation method to get the Nonlinear Schrodinger (NLS) equation with variable coefficient. Bi, Zhang, Liu and Liu [20] studied the propagation of nonlinear blood flow in artery by using radial basis function method to obtain the higher order of nonlinear Schrödinger equation.

From those literature reviews, researchers concluded that the study of wave propagation in the stenosed elastic tube with variation viscosity of a Newtonian fluid is worthy to carry out because it can detect the abnormal arteries in the human body through the changes of characteristics of nonlinear waves during propagation [21]. Therefore, for present research, the artery is assumed as a prestressed elastic tube with thin wall with symmetrical stenosis and treating the blood as an incompressible Newtonian fluid with variable viscosity. Besides, the propagation of solitary waves in this medium is investigated by applying the reductive perturbation method. There are three objectives that are concerned in this study. Firstly, to derive the nonlinear partial differential equation for wave propagation in a Newtonian fluid with variable viscosity filled in the prestressed thin-walled stenosed elastic tube by using reductive perturbation method. Next, determine the progressive wave solution for nonlinear partial differential equation. Then, analyse the solution of progressive wave on the variation of wave speed, fluid pressure function and the radial displacement in the presence and absence of the stenosis, respectively.

2. Basic Equations and Methods

2.1 Equation of Tube

The artery is a tube with three layers of tissues: intima, tunica media, and adventitia. For the present study, the artery is identified as a elastic tube with stenosed wall. The mathematical model for this study is illustrated as in Figure 1, where r_0 represents the deformed radius at the coordinate system's origin, z^* is the axial coordinate after static deformation, $f(z^*)$ is a function that characterizes the axially symmetric bump on the surface of the arterial wall and u^* is a dynamical radial displacement. The equation of motion of the tube in the radial direction as in the Eq. 1 [13], where μ is the shear modulus, H is the thickness of tube material, $R^*(z^*)$ is the radius of circular cross-section tube, $r^*(z^*)$ characterizes the variable radius after this static deformation, Σ is the strain energy density function of the membrane, λ_1 is the stretch ratios along the meridional curves, λ_z is an initial axial stretch ratio in the arteries direction, λ_2 is the stretch ratios along the circumferential curves, P_r^* is the radial fluid reaction forces on the initial surface of the tube, ρ_0 is the mass density of the membrane material, t^* is the time parameter, and Λ is defined by $\left[1+\left(r^{*\prime}+\frac{\partial u^*}{\partial z^*}\right)^2\right]^{\frac{1}{2}}$. The tube's equation of motion in the redial direction is as follows:

$$\frac{\partial}{\partial z^*}\left\{\frac{\mu H R^*(z^*)}{\Lambda}\left(r^{*\prime}+\frac{\partial u^*}{\partial z^*}\right)\frac{\partial \Sigma}{\partial \lambda_1}\right\}-\frac{\mu H}{\lambda_z}\left[1+\lambda_z^2\left(R^{*\prime}\right)^2\right]^{1/2}\frac{\partial \Sigma}{\partial \lambda_2}+\Lambda\left(r^*+u^*\right)P_r^*-\rho_0\frac{H}{\lambda_z}R^*\left(z^*\right)$$

$$\times\left[1+\lambda_z^2\left(R^{*\prime}\right)^2\right]^{1/2}\frac{\partial^2 u^*}{\partial t^{*2}}=0. \qquad Eq.1$$

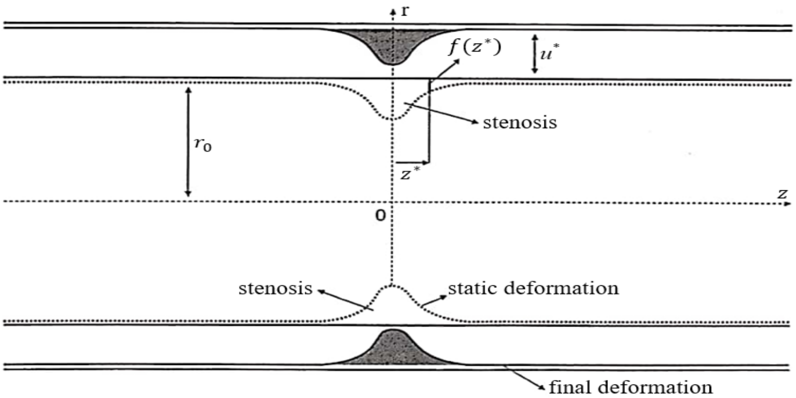

Figure 1: The geometry of the tube [13].

2.3 Equations of fluids

The blood flow inside the artery is assumed to be an incompressible Newtonian fluid with variable viscosity. Because it can violate the non-slip condition at the boundary by vanishing the viscosity on the wall of artery, its maximum value can be reached at the artery's center, which satisfies the assumption for the flow problems in large arteries [13]. The incompressible Newtonian fluid with variable coefficient whose axially symmetric motion in the cylindrical polar coordinates can be expressed as [13]:

$$\frac{\partial V_r^*}{\partial t^*}+V_r^*\frac{\partial V_r^*}{\partial r}+V_z^*\frac{\partial V_r^*}{\partial z^*}+\frac{1}{\rho_f}\frac{\partial \overline{P}}{\partial r}-2\hat{v}\gamma'(r)\frac{\partial V_r^*}{\partial r}-\hat{v}\gamma(r)\left(\frac{\partial^2 V_r^*}{\partial r^2}+\frac{1}{r}\frac{\partial V_r^*}{\partial r}-\frac{V_r^*}{r^2}+\frac{\partial^2 V_r^*}{\partial z^{*2}}\right)=0, \qquad Eq.2$$

$$\frac{\partial V_z^*}{\partial t^*}+V_r^*\frac{\partial V_z^*}{\partial r}+V_z^*\frac{\partial V_z^*}{\partial z^*}+\frac{1}{\rho_f}\frac{\partial \overline{P}}{\partial z^*}-\hat{v}\gamma'(r)\left(\frac{\partial V_r^*}{\partial z^*}+\frac{\partial V_z^*}{\partial r}\right)-\hat{v}\gamma(r)\left(\frac{\partial^2 V_z^*}{\partial r^2}+\frac{1}{r}\frac{\partial V_z^*}{\partial r}+\frac{\partial^2 V_z^*}{\partial z^{*2}}\right)=0, \qquad Eq.3$$

$$\frac{\partial V_r^*}{\partial r}+\frac{V_r^*}{r}+\frac{\partial V_r^*}{\partial z^*}=0 \ \text{(incompressibility)}, \qquad Eq.4$$

with the boundary conditions

$$V_r^*\Big|_{r=r_f}=\frac{\partial u^*}{\partial t^*}+\left(r^{*\prime}+\frac{\partial u^*}{\partial z^*}\right)V_z^*\Big|_{r=r_f}, \qquad Eq.5$$

$$P_r^*=\frac{1}{\Lambda}\left[\overline{P}-2\rho_f\hat{v}\gamma(r)\frac{\partial V_r^*}{\partial r}+\rho_f\hat{v}\gamma(r)\left(r^{*\prime}+\frac{\partial u^*}{\partial z^*}\right)\left(\frac{\partial V_r^*}{\partial z^*}+\frac{\partial V_z^*}{\partial r}\right)\right]\Big|_{r=r_f}, \qquad Eq.6$$

where V_r^* and V_z^* are the radial components and the axial velocity components of the fluid body, respectively. ρ_f is the mass density of the fluid, $\hat{v}$ is the kinematical viscosity of the fluid at the tube's center and $y(r)$ is the variation of viscosity, P_r^* is the fluid reaction force density and $r_f = r^*(z^*) + u^*$.

2.4 Nondimensionalized Equations

The dimensionless quantities are used to obtain the equations of tube and fluid without dimension from their dimensional equations. Thus, the following dimensionless quantities are introduced [22]:

$$t^* = (R_0/c_0)t, \quad z^* = R_0 z, \quad u^* = R_0 u, \quad V^*_r = c_0 V_r, \quad V^*_z = c_0 V_z, \quad \overline{P} = \rho_f c_0^2 \overline{p}, \quad \hat{v} = c_0 R_0 \overline{v}, \quad r = R_0 x,$$
$$R^*(z^*) = R_0[1 - F(z)], \quad m = \rho_0 H/\rho_f R_0, \quad r^*(z^*) = R_0[\lambda_\theta - f(z)], \quad c_0^2 = \mu H/\rho_f R_0,$$
$$P^*_r = \frac{1}{\wedge}\rho_f c_0^2 P_r, \quad \wedge = \left[1 + \left(r^{*'} + \partial u^*/\partial z^*\right)^2\right]^{1/2}, \qquad Eq.7$$

where c_0 is a Moens-Korteweg wave speed, R_0 known as initial reference radius, $\lambda_\theta = r_0/R_0$ is defined as the initial stretch ratio.

Utilizing Eq. 7 into the Eq. 1 – Eq. 6 and using the chain rule, the non-dimensional equations are achieved as below:

$$P_r = \frac{m[1 - F(z)]\left[1 + \lambda_z^2 (F')^2\right]^{1/2}}{\lambda_z(\lambda_\theta - f(z) + u)} \frac{\partial^2 u}{\partial t^2} + \frac{\left[1 + \lambda_z^2 (F')^2\right]^{1/2}}{\lambda_z(\lambda_\theta - f(z) + u)} \frac{\partial \Sigma}{\partial \lambda_2} - \frac{1}{(\lambda_\theta - f(z) + u)} \frac{\partial}{\partial z}$$
$$\times \left\{[1 - F(z)]\left(-f' + \frac{\partial u}{\partial z}\right) \Big/ \left[1 + \left(-f' + \frac{\partial u}{\partial z}\right)^2\right]^{1/2}\right\} \frac{\partial \Sigma}{\partial \lambda_1}. \qquad Eq.8$$

$$\frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial x} + V_z \frac{\partial V_r}{\partial z} + \frac{\partial \overline{p}}{\partial x} - 2\overline{v}\gamma'(x)\frac{\partial V_r}{\partial x} - \overline{v}\gamma(x)\left(\frac{\partial^2 V_r}{\partial x^2} + \frac{1}{x}\frac{\partial V_r}{\partial x} - \frac{V_r}{x^2} + \frac{\partial^2 V_r}{\partial z^2}\right) = 0, \qquad Eq.9$$

$$\frac{\partial V_z}{\partial t} + V_r \frac{\partial V_z}{\partial x} + V_z \frac{\partial V_z}{\partial z} + \frac{\partial \overline{p}}{\partial z} - \overline{v}\gamma'(x)\left(\frac{\partial V_r}{\partial z} + \frac{\partial V_z}{\partial x}\right) - \overline{v}\gamma(x)\left(\frac{\partial^2 V_z}{\partial x^2} + \frac{1}{x}\frac{\partial V_z}{\partial x} + \frac{\partial^2 V_z}{\partial z^2}\right) = 0, \qquad Eq.10$$

$$\frac{\partial V_r}{\partial x} + \frac{V_r}{x} + \frac{\partial V_z}{\partial z} = 0, \qquad Eq.11$$

with boudary conditions

$$V_r\Big|_{x=\lambda_\theta - f(z)+u} = \frac{\partial u}{\partial t} + \left(-f' + \frac{\partial u}{\partial z}\right)V_z\Big|_{x=\lambda_\theta - f(z)+u}, \qquad Eq.12$$

$$P_r = \left[\overline{p} - 2\overline{v}\gamma(x)\frac{\partial V_r}{\partial z} + \overline{v}\gamma(x)\left(-f' + \frac{\partial u}{\partial z}\right)\left(\frac{\partial V_r}{\partial z} + \frac{\partial z}{\partial x}\right)\right]\Big|_{x=\lambda_\theta - f(z)+u}. \qquad Eq.13$$

3. Long wave approximation

This study examined small-but-finite amplitude solitary waves propagation in a prestressed thin-walled elastic stenosed tube filled with the incompressible Newtonian fluid with variable viscosity. Then, the reductive perturbation method will be used. The stretched coordinates are introduced for the boundary-value problem [22]:

$$\xi = \varepsilon^{\frac{1}{2}}(z - ct), \quad \tau = \varepsilon^{\frac{1}{2}} z, \qquad Eq.14$$

where ε denotes a small parameter measuring the weakness of nonlinearity and dispersion, and c is the scale parameter to be determined from the solution. $G(z)$ and $g(z)$ must be of orders 5/2 to consider the stenosis's effect in the nonlinear differential equations. Thus, G and g are assumed to have the following form [18]:

$$\hat{G}(\xi,\tau) = \varepsilon G(\tau), \quad \hat{g}(\xi,\tau) = \varepsilon g(\tau). \qquad Eq.15$$

Eq. 16 is the type of differential relations. It introduces all the equations of tube and fluid in dimensionless form.

$$\partial/\partial t = -\varepsilon^{1/2} c\left(\partial/\partial\xi\right), \qquad \partial/\partial z = \varepsilon^{1/2}\left[\partial/\partial\xi + \varepsilon\left(\partial/\partial\tau\right)\right]. \qquad Eq.16$$

For the long wave limit, it considered that the field variables are expanded into asymptotic series of ε as the following form [22]:

$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \cdots, \quad \gamma(x) = \gamma_0(x) + \varepsilon\gamma_1(x) + \varepsilon^2\gamma_2(x) + \cdots, \quad V_r = \varepsilon^{1/2}\left(\varepsilon V_r^{(1)} + \varepsilon^2 V_r^{(2)} + \cdots\right),$$
$$V_z = \varepsilon V_z^{(1)} + \varepsilon^2 V_z^{(2)} + \cdots, \quad P_r = P_r^{(0)} + \varepsilon P_r^{(1)} + \varepsilon^2 P_r^{(2)} + \cdots, \quad \overline{p} = \overline{p}_0 + \varepsilon\overline{p}_1 + \varepsilon^2\overline{p}_2 + \cdots. \qquad Eq.17$$

where,

$$\gamma_0(x) = 1 - x/\lambda_\theta, \ \ \gamma_1(x) = \left(x/\lambda_\theta^2\right)\left(u_1 - g\right), \ \ \gamma_2(x) = \left(x/\lambda_\theta^2\right)\left[u_2 - \left(u_1 - g\right)^2/\lambda_\theta\right]. \qquad Eq.18$$

By introducing Eq. 16 and Eq. 17 into Eq. 9 until Eq. 13 yield the following various order of nonlinear differential equations.

$O(\varepsilon)$ equations:

$$\frac{\partial\overline{p_1}}{\partial x} = 0, \qquad \frac{\partial V_r^{(1)}}{\partial x} + \frac{V_r^{(1)}}{x} + \frac{\partial V_z^{(1)}}{\partial\xi} = 0,$$
$$-c\frac{\partial V_z^{(1)}}{\partial\xi} + \frac{\partial\overline{p_1}}{\partial\xi} - \nu\gamma_0'(x)\frac{\partial V_z^{(1)}}{\partial x} - \nu\gamma_0(x)\left(\frac{\partial^2 V_z^{(1)}}{\partial x^2} + \frac{1}{x}\frac{\partial V_z^{(1)}}{\partial x}\right) = 0, \qquad Eq.19$$

with boundary conditions

$$V_r^{(1)}\Big|_{x=\lambda_\theta} = -c\left(\partial u_1 \ /\partial\xi\right), \quad \overline{p_1}\Big|_{x=\lambda_\theta} = P_r^{(1)}. \qquad Eq.20$$

$O(\varepsilon^2)$ equations:

$$-c\frac{\partial V_r^{(1)}}{\partial\xi} + \frac{\partial\overline{p_2}}{\partial x} - 2\nu\gamma_0'(x)\frac{\partial V_r^{(1)}}{\partial x} - \nu\gamma_0(x)\left[\frac{\partial^2 V_r^{(1)}}{\partial x^2} + \frac{1}{x}\frac{\partial V_r^{(1)}}{\partial x} - \frac{V_r^{(1)}}{x^2}\right] = 0,$$
$$-c\frac{\partial V_z^{(2)}}{\partial\xi} + V_r^{(1)}\frac{\partial V_z^{(1)}}{\partial x} + V_z^{(1)}\frac{\partial V_z^{(1)}}{\partial\xi} + \frac{\partial\overline{p_2}}{\partial\xi} + \frac{\partial\overline{p_1}}{\partial\tau} - \nu\gamma_0'(x)\left[\frac{\partial V_r^{(1)}}{\partial\xi} + \frac{\partial V_z^{(2)}}{\partial x}\right] - \nu\gamma_1'(x)\frac{\partial V_z^{(1)}}{\partial x}$$
$$-\nu\gamma_0(x)\left[\frac{\partial^2 V_z^{(2)}}{\partial x^2} + \frac{1}{x}\frac{\partial V_z^{(2)}}{\partial x} + \frac{\partial^2 V_z^{(1)}}{\partial\xi^2}\right] - \nu\gamma_1(x)\left[\frac{\partial^2 V_z^{(1)}}{\partial x^2} + \frac{1}{x}\frac{\partial V_z^{(1)}}{\partial x}\right] = 0,$$
$$\frac{\partial V_r^{(2)}}{\partial x} + \frac{V_r^{(2)}}{x} + \frac{\partial V_z^{(2)}}{\partial\xi} + \frac{\partial V_z^{(1)}}{\partial\tau} = 0, \qquad Eq.21$$

with boundary conditions

$$\left[V_r^{(2)} + [u_1 - g(\tau)]\frac{\partial V_r^{(1)}}{\partial x}\right]\Big|_{x=\lambda_\theta} = -c\frac{\partial u_2}{\partial \xi} + \frac{\partial u_1}{\partial \xi}V_z^{(1)}\Big|_{x=\lambda_\theta},$$

$$P_r^{(2)} = \left[\overline{p_2} + [u_1 - g(\tau)]\frac{\partial \overline{p_1}}{\partial x} - 2v\gamma_0(x)\frac{\partial V_r^{(1)}}{\partial x}\right]\Big|_{x=\lambda_\theta}. \qquad Eq.22$$

Thus, the viscosity is assumed as the order of $\varepsilon^{\frac{1}{2}}$, then $\underline{v} = \varepsilon^{\frac{1}{2}}v$.

The expressions of $Pr^{(1)}$ and $Pr^{(2)}$ with the radial displacement, u must know to complete the equations.

$$P_r^{(1)} = \beta_1(u_1 - g) + \gamma_1\lambda_\theta G,\ P_r^{(2)} = \left(mc^2/\lambda_\theta\lambda_z - \beta_0\right)\left(\partial^2 u_1/\partial\xi^2\right) + \beta_1 u_2 + \beta_2 u_1^2 + \beta_3(\tau)u_1 + \pi(\tau), \quad Eq.23$$

where

$$\beta_3(\tau) = 2[\lambda_\theta\gamma_2 G - \beta_2 g], \qquad \pi(\tau) = \left(\lambda_\theta^2\gamma_2 + \lambda_\theta\gamma_1\right)G^2 + \beta_2 g^2 - 2\gamma_2\lambda_\theta G g. \qquad Eq.24$$

The coefficients of $\gamma_0, \gamma_1, \gamma_2, \beta_0, \beta_1$ and β_2 are defined as below

$$\gamma_0 = (1/\lambda_\theta\lambda_z)(\partial\Sigma/\partial\lambda_\theta), \quad \gamma_1 = (1/\lambda_\theta\lambda_z)\left(\partial^2\Sigma/\partial{\lambda_\theta}^2\right), \quad \gamma_2 = (1/2\lambda_\theta\lambda_z)\left(\partial^3\Sigma/\partial{\lambda_\theta}^3\right),$$

$$\beta_0 = (1/\lambda_\theta)(\partial\Sigma/\partial\lambda_z), \quad \beta_1 = \gamma_1 - \gamma_0/\lambda_\theta, \quad \beta_2 = \gamma_2 - \beta_1/\lambda_\theta. \qquad Eq.25$$

3.1 Solution of the Field Equations

At this stage, the solutions for the various order of differential equations Eq. 19 – Eq. 22 are determined in order to obtain the governing equation for the corresponding mathematical model. From the solution of Eq. 19 and Eq. 23 under the boundary conditions Eq. 20, one can get

$$u_1 = U(\xi,\tau), \quad \overline{p_1} = \beta_1(U - g) + \gamma_1\lambda_\theta G, \quad V_z^{(1)} = (\beta_1/c)U, \quad V_r^{(1)} = -(\beta_1/v)(\partial U/\partial\xi)x, \qquad Eq.26$$

where c is known as phase velocity in the long wave approximation, and the following condition is obtained $\beta_1 = 2c^2/\lambda_\theta$ and $U(\xi,\tau)$ is known as unknown function that governing equation will be determined afterward.

Using the solution of Eq. 26 into the Eqs. 21 – 23 and by eliminating the u_2, then, the following governing equation is achieved: forced Kortweg-de Vries Burger (FKdVB) equation with variable coefficient.

$$\frac{\partial U}{\partial \tau} + \mu_1 U\frac{\partial U}{\partial \xi} - \mu_2\frac{\partial^2 U}{\partial \xi^2} + \mu_3\frac{\partial^3 U}{\partial \xi^3} + \mu_4(\tau)\frac{\partial U}{\partial \xi} = \mu(\tau), \qquad Eq.27$$

where the coefficients are defined by

$$\mu_1 = \frac{\beta_2}{\beta_1} + \frac{5}{2\lambda_\theta}, \quad \mu_2 = \frac{v}{2c}, \quad \mu_3 = \frac{m}{4\lambda_z} - \frac{\beta_0}{2\beta_1} + \frac{{\lambda_\theta}^2}{16}, \quad \mu_4(\tau) = \frac{\lambda_\theta\gamma_2}{\beta_1}G(\tau) - \left[\frac{\beta_2}{\beta_1} + \frac{1}{2\lambda_\theta}\right]g(\tau),$$

$$\mu = \frac{1}{2}g'(\tau) - \frac{\gamma_1\lambda_\theta}{2\beta_1}G'(\tau). \qquad Eq.28$$

3.2 Progressive Waves Solution

The analytical solution to the FKdVB problem with variable coefficients was investigated in this section. To begin, consider a new dependent variable, *V* as follow [22]:

$$U(\xi,\tau)=V(\xi,\tau)+\int_0^\tau \mu(s)\,ds=V(\xi,\tau)+(1/2)\left[g(\tau)-(\lambda_\theta\gamma_1/\beta_1)G(\tau)\right]. \qquad Eq.29$$

Introducing Eq. 29 into Eq. 27 yields

$$\frac{\partial V}{\partial\tau}+\mu_1 V\frac{\partial V}{\partial\xi}-\mu_2\frac{\partial^2 V}{\partial\xi^2}+\mu_3\frac{\partial^3 V}{\partial\xi^3}+\left[\frac{\mu_1}{2}\left(g(\tau)-\frac{\lambda_\theta\gamma_1}{\beta_1}G(\tau)\right)+\mu_4(\tau)\right]\frac{\partial V}{\partial\xi}=0. \qquad Eq.30$$

Then, introduce the coordinate transformation as follow

$$\tau'=\tau,\quad \xi'=\xi-\int_0^\tau\left\{(\mu_1/2)\left[g(s)-(\lambda_\theta\gamma_1/\beta_1)G(s)\right]+\mu_4(s)\right\}ds. \qquad Eq.31$$

By introducing Eq. 31 into Eq. 30, the FKdVB equation with variable coefficient reduces to the KdVB equation

$$\partial V/\partial\tau'+\mu_1 V(\partial V/\partial\xi')-\mu_2\left(\partial^2 V/\partial\xi'^2\right)+\mu_3\left(\partial^3 V/\partial\xi'^3\right)=0. \qquad Eq.32$$

The solution of the KdVB equation is given by [22]

$$V=a/\mu_1+\left(3\mu_2^2/25\mu_1\mu_3\right)\left(\operatorname{sech}^2\zeta-2\tanh\zeta\right), \qquad Eq.33$$

where the term *a* is known as a constant and ζ is a phase function defined by

$$\zeta=(\mu_2/10\mu_3)(\xi'-a\tau'). \qquad Eq.34$$

Inserting Eq. 28, Eq. 28, Eq. 33 and Eq. 34 into Eq. 29, the analytical solution of the evolution equation Eq. 27 is obtained as

$$U=\frac{a}{\mu_1}+\frac{3\mu_2^2}{25\mu_1\mu_3}\left(\operatorname{sech}^2\zeta-2\tanh\zeta\right)+\frac{1}{2}\left[g(\tau)-\frac{\lambda_\theta\gamma_1}{\beta_1}G(\tau)\right], \qquad Eq.35$$

where

$$\zeta=\frac{\mu_2}{10\mu_3}\left(\xi-a\tau-\int_0^\tau\left[\left(\frac{3}{4\lambda_\theta}-\frac{\beta_2}{2\beta_1}\right)g(s)+\frac{\lambda_\theta}{\beta_1}\left(\gamma_2-\frac{\mu_1\gamma_1}{2}\right)G(s)\right]ds\right). \qquad Eq.36$$

The fluid pressure, $\underline{p_1}$ is obtained by substituting the Eq. 35 into Eq. 26, one can get

$$\overline{p_1}=a\beta_1/\mu_1+\left(3\beta_1\mu_2^2/25\mu_1\mu_3\right)\left(\operatorname{sech}^2\zeta-2\tanh\zeta\right)-(\beta_1/2)g(\tau)+(\lambda_\theta\gamma_1/2)G(\tau). \qquad Eq.37$$

The wave speed is given by,

$$\upsilon_p=\frac{d\tau}{d\xi}=1\bigg/\left\{a+\left(\frac{3}{4\lambda_\theta}-\frac{\beta_2}{2\beta_1}\right)g(\tau)+\frac{\lambda_\theta}{\beta_1}\left(\gamma_2-\frac{\mu_1\gamma_1}{2}\right)G(\tau)\right\}. \qquad Eq.38$$

3.3 Numerical Results and Discussion

The strain energy density function, Σ and the coefficients of $\gamma_0, \gamma_1, \gamma_2, \beta_0, \beta_1$ and β_2 are given by [18]. Demiray [18] stated that using the value of material constant, α = 1.948 and the value of axial stretch of the artery and circumferential stretch $\lambda_z = \lambda_\theta$ = 1.6, can get γ_0 = 49.183, γ_1 = 326.844, γ_2 = 1176.561, β_0 = 78.692, β_1 = 296.105, β_2 = 991.496, c = 15.391, μ_1 = 4.911, μ_2 = 0.0325, μ_3 = 0.043, provided m = 0.1 and v = 1. The coefficient of $\mu_4(\tau)$ and $\mu(\tau)$ can be obtained through specifying the functions $G(\tau)$ and $g(\tau)$ which characterize the stenosis's shape in the deformed and undeformed states. Then, set $G(\tau)$ = 0, $g(\tau) = sech(\delta\tau)$ and a = 1.

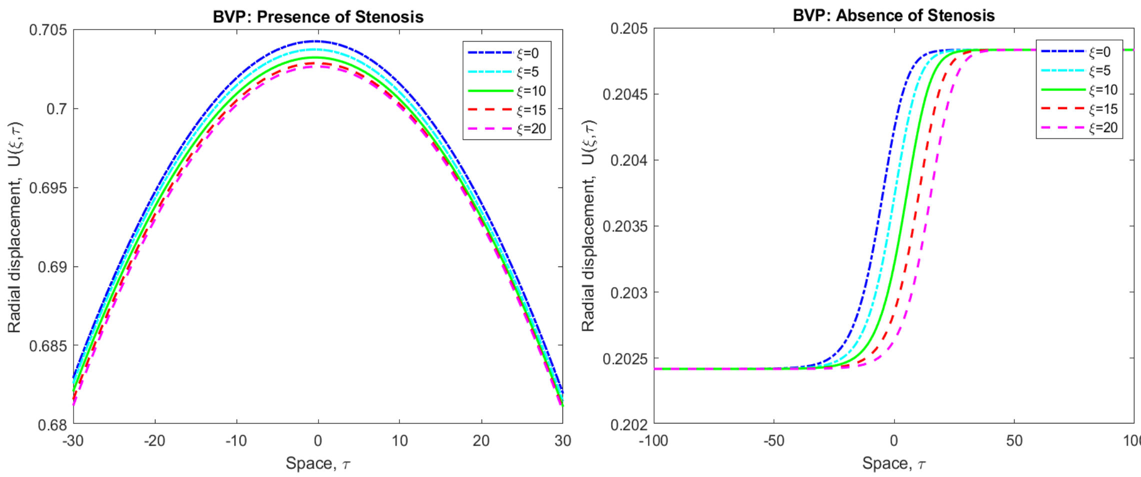

Figure 2: BVP: Radial displacement U versus space τ for a different time at δ = 0.01 in the presence of stenosis (i) and the absence of stenosis (ii), respectively.

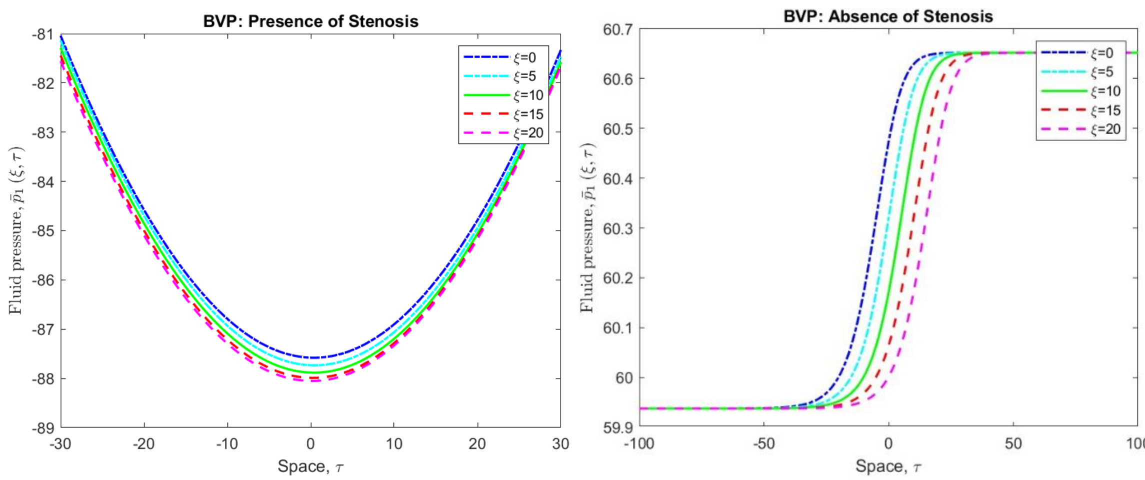

Figure 3: BVP: Fluid pressure function versus space τ for different time ξ at δ = 0.01 in the presence of stenosis (i) and the absence of stenosis (ii), respectively.

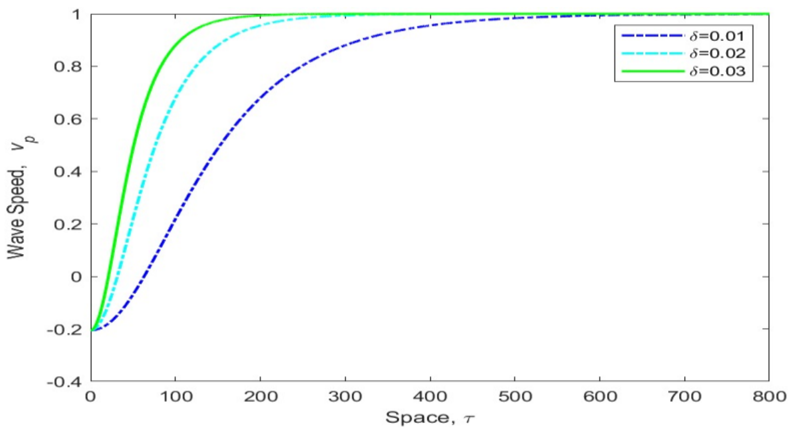

Figure 4: The wave speed, v_p for different δ

Figure 2 illustrates the results for the radial displacement U in the existence of stenosis (the solution of the FKdVB equation with variable coefficient) and the absence of stenosis (the solution of the KdVB equation with variable coefficient), respectively. Figure 2(i) shows that the amplitude of the bell-shaped wave decreases as time increases. At space τ = 0, the differences between two lines plotted for the values of U from the top are 5.17×10^{-4}, 5.04×10^{-4}, 3.61×10^{-4} and 2.12×10^{-4}. The difference between two lines becomes smaller and smaller from the top. Figure 2(ii) shown an increasing shock wave profile where this shock wave profile propagates towards right as time increases. From these two figures, it is observed that the solution of FKdVB equation with variable coefficient gives a solitary wave, but the solution of the KdVB equation gives a shock wave profile. Hence, it can be concluded that the variable viscosity in fluid plays an important role to determine the blood flow characteristic. The solution of the fluid pressure function with space τ in the presence of stenosis and absence of stenosis are plotted in Figures 3(i) and 3(ii), respectively. The inverse solitary wave profile as in Figure 3(i) is because of variable viscosity and stenosis. Besides, due to the presence of stenosis, the fluid pressure function reached a minimal value at the stenosis's core. By discarding the stenotic effect, an increasing shock wave profile propagates to the right as time increases, as seen in Figure 3(ii). Next, the effects of the severity of the stenosis on the wave speed has been presented in Figure 4. Figure 4 shows the wave speed of the FKdVB equation increases with space τ till a terminal velocity. Besides, it shows as δ increases, the wave speed increases faster. As seen in Eq. 38, due to the presence of stenosis, the wave speed fluctuates along the tube axis.

4. Conclusion

In conclusion, the propagation of solitary wave in a stenosed thin elastic tube filled Newtonian fluid with variable viscosity is studied by using the reductive perturbation method. The evolution equation for current research is the forced Korteweg-de Vries-Burgers (FKdVB) equation with variable coefficient. From the graphical outputs, it is concluded that in the absence of stenosis, the radial displacement and fluid pressure function show increasing shock wave profile propagates to the right by preserving their form as time increases. On the other hand, in the presence of stenosis, radial displacement shows bell-shape wave, but fluid pressure function shows inverse bell-shaped wave. All these amplitudes of waves decrease as time increases. From graphical output of wave speed, it shows an increasing wave speed until a terminal velocity 1 is achieved.

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