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Milne-Simpson Method for Solving First Order Fuzzy Differential Equations Using Hukuhara Approach

Mazarina Mohd Noor[1], Syahirbanun Isa[1*]

[1]Department of Mathematics and Statistics, Faculty of Applied Sciences and Technology,
Universiti Tun Hussein Onn Malaysia, 84600 Pagoh, Johor, MALAYSIA

*Corresponding Author Designation

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Abstract: This paper discusses the linear multistep method for solving fuzzy differential equations. The Hukuhara technique will be used to convert the fuzzy differential equations into a system of ordinary differential equations. To solve first-order fuzzy differential equations, the Milne-Simpson method is used. The solutions are compared with Runge-Kutta method. Numerical examples are given, and numerical solutions are displayed to present the accuracy of the method.

Keywords: Fuzzy Differential Equations, Hukuhara Approach, Milne-Simpson Method

1. Introduction

In recent years, the field of fuzzy differential equations (FDEs) has gained in popularity. The theory of fuzzy derivative was initially developed by Dubois & Prade, then followed by Chang & Zadeh, who used the extension principle in their technique [1]. Dubois and Prade employs the derivative in the case of fuzzy-set-valued functions in which each $t \in [a,b]$ is associated with a fuzzy integer [2]. Puri & Ralescu was the first to introduce such derivatives, which were derived from the Hukuhara derivative for real-valued functions [3]. This derivative is the source of the most widely used and explored FDEs theory. The problem with this form of differential equations is that the solutions grow in diameter over time. This implies that as time goes on, the process becomes increasingly vague. Bede & Gal improved on Puri's concept of derivatives in such a manner that the diameter of the solutions of FDEs does not have to increase [4]. That is to say; the process can get less vague over time.

The second form of FDEs theory employs fuzzy sets of functions rather than fuzzy-set-valued functions. Fuzzy sets of functions will be considered as fuzzy functions. Hüllermeier introduced their theoretical foundation as differential inclusions theory and fuzzy differential inclusions theory [5]. Because it separates the functions of the support of fuzzy sets of functions, there is no concept of a

*Corresponding author: syahir@uthm.edu.my

fuzzy function's derivative in this type of FDEs and the derivative used is the same as for standard functions.

When there are unknown parameters or starting conditions, the solution of FDEs is critical. This uncertainty can be expressed as a fuzzy number. The numerical method can be utilized to solve the problem when the variables contain uncertainty information. To solve FDEs using numerical methods, the stable approach is created. It is difficult to find a precise solution for fuzzy initial value problems (FIVPs). FIVP shows up when the modeling of these issues was defective or not clear and its nature is beneath vulnerability [6]. FIVPs do not have a derivative of a fuzzy –number-valued function, and so, the numerical solutions of a FDE are difficult to be obtained. As a result, their numerical technique has to be considered.

The FIVP is written as follows

$$y'(t) = f(t, y(t)), \qquad Eq. 1$$
$$y(t_0) = y_0, \;\; t \in [t_0, T]$$

Milne – Simpson method is one of the methods in the linear multistep method (LMM). LMM is one of the important methods for solving numerical solution of FDEs. This multistep approach uses approximation at several previous mesh locations to derive the approximation at the following locations [7]. Meanwhile, the one-step method or single-step method refers to a value of the dependent variable at one mesh location that is required to compute the value at the next mesh location. As a result, as compared to single-step procedures, the multistep method tries to increase efficiency and more accurate outcomes. Generally, multistep method is more accurate and efficient compared to single-step method [8], [9], [10].

The general LMM equation is as follows:

$$\sum_{k=0}^{i} \alpha_k y_{n+k} = h \sum_{k=0}^{i} \beta_k f_{n+k} \qquad Eq. 2$$

where α_k and β_k are constant, assume $\alpha_k \neq 0$ and $\alpha_k \neq \beta_k \neq 0$.

There are two parts to the multistep method:

i. when the $\beta_k \neq 0 \Rightarrow$ implicit multistep method
ii. when the $\beta_k = 0 \Rightarrow$ explicit multistep method

In the Milne-Simpson method, the explicit multistep method called Milne's method is used as a predictor to an implicit multistep method called Simpson's method.

The objectives of this study are to transform the FDEs into the system of ordinary differential equations (ODEs) by using the Hukuhara approach. Then, solve the first-order FDEs by using the Milne – Simpson multistep method and compare the solution with the Runge – Kutta (RK) method.

2. Research Methodology

The focus of this study is to solve the first order FDEs using Hukuhara approach. Then, the problem will be solved by using Milne-Simpson method.

2.1 Hukuhara Approach

To find a precise solution to the fuzzy initial value problem, Hukuhara differentiability (H-derivative) is applied. The concept of Hukuhara in Definition 1 and 2 as presented in [11].

Definition 1

Let $x, y \in \mathrm{E}$. If $\exists\, z \in \mathrm{E}$ such that $x = y \oplus z$,.then z is called the Hukuhara differentiability of x and y, it is denoted by $!\ y$. Hukuhara differentiability is represented by the symbol "! " and note that $x\,!\ y \neq x + (-1)y$.

Definition 2

Let $f : \Box \rightarrow \mathrm{E}$ be a fuzzy function. Then, f is differentiable $t_0 \in \Box$ if $\exists f'(t_0) \in \mathrm{E}$ such that

$$\lim_{h \to 0^+} \frac{f(t_0 + h)\,!\ \ f(t_0)}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{f(t_0)\,!\ \ f(t_0 - h)}{h} \qquad Eq.3$$

exist and are equivalent to $f'(t_0)$.

Here the limits are taken in the metric space (E, A), since we have defined

$$h^{-1} \Box\ (f(t_0)\,!\ \ f(t_0 - h)) \text{ and } h^{-1} \Box\ (f(t_0 + h)\,!\ \ f(t_0)) \qquad Eq.4$$

where $\Box$ represent for Hadamard product (component-wise multiplication for matrices).

Consider FIVP in Eq. 1 and let $y(t) = [\underline{y}(t), \overline{y}(t)]$, and consider $y(t)$ is Hukuhara differentiable, Then,

$$y'(t) = [\underline{y}'(t), \overline{y}'(t)] \qquad Eq.5$$

and

$$f(t, y(t)) = \left[\underline{f}\left(t, \underline{y}(t), \overline{y}(t)\right), \overline{f}\left(t, \underline{y}(t), \overline{y}(t)\right)\right] \qquad Eq.6$$

Therefore, FIVP may be written in the first order ODE system,

$$\begin{gathered} \underline{y}'(t) = \underline{f}\left(t, \underline{y}(t), \overline{y}(t)\right), \\ \overline{y}'(t) = \overline{f}\left(t, \underline{y}(t), \overline{y}(t)\right), \\ \underline{y}(t_0) = \underline{y}_0, \\ \overline{y}(t_0) = \overline{y}_0 \end{gathered} \qquad Eq.7$$

This has a unique solution $(\underline{y}, \overline{y})$.

The parametric form of Eq. 7 is given by

$$\begin{gathered} \underline{y}'(t;r) = \underline{f}\left(t, \underline{y}(t;r), \overline{y}(t;r)\right), \\ \overline{y}'(t;r) = \overline{f}\left(t, \underline{y}(t;r), \overline{y}(t;r)\right), \\ \underline{y}(t_0;r) = \underline{y}_0(r), \\ \overline{y}(t_0;r) = \overline{y}_0(r), \text{ for } r \in [0,1]. \end{gathered} \qquad Eq.8$$

2.2 Milne-Simpson Methods

The Milne-Simpson method is a well-known corrector-predictor approach. Milne's method functions as a predictor, whereas Simpson's method functions as a corrector.

The general formula of Milne's method is written as follow

$$y_{n+1} = y_{n-3} + \frac{4}{3}h\left(2f_n - \frac{4}{3}f_{n-1} + 2f_{n-2}\right) \qquad Eq.9$$

and the general formula of Simpson's method is written as follow

$$y_{n+1} = y_{n-1} + \frac{h}{3}\left(f_{n+1} - 4f_n + f_{n-1}\right) \qquad Eq.10$$

2.3 Fuzzy Formulation of Milne-Simpson Method

In this study, Milne – Simpson method will be transformed to fuzzy terms using the Hukukara approach as defined in Section 2.1. The fuzzy formulation of Milne – Simpson method will then be used to solve the problems of FDEs. For t as a positive integer, approximate the solution of FIVP in Eq. 1.

Step 1
Let $h=\frac{T-t_0}{N}$,

Step 2
The Runge-Kutta method is used to find the two starting points after the initial is obtained from the FIVP.

$$\underline{y}(t_0;r)=\underline{y}_0 \quad \underline{y}(t_1;r)=\underline{y}_1 \quad \underline{y}(t_2;r)=\underline{y}_2$$
$$\overline{y}(t_0;r)=\overline{y}_0 \quad \overline{y}(t_1;r)=\overline{y}_1 \quad \overline{y}(t_2;r)=\overline{y}_2 \qquad Eq.11$$

By using Runge-Kutta method

$$\underline{K}_1=h\underline{f}\left(t_1,\underline{y}_1(r),\overline{y}_1(r)\right)$$
$$\underline{K}_2=h\underline{f}\left(t_1+\frac{h}{2},\underline{y}_1(r),\overline{y}_1(r)+\frac{\underline{K}_1}{2}\right)$$
$$\underline{K}_3=h\underline{f}\left(t_2+\frac{h}{2},\underline{y}_2(r),\overline{y}_2(r)+\frac{\underline{K}_2}{2}\right)$$
$$\underline{K}_4=h\underline{f}\left(t_3+h,\underline{y}_3(r),\overline{y}_3(r)+h\right)$$
$$\underline{y}_{i+1}=\underline{y}_i+\frac{1}{6}\left(\underline{K}_1+2\underline{K}_2+2\underline{K}_3+\underline{K}_4\right)$$

$$\overline{K}_1=h\overline{f}\left(t_1,\underline{y}_1(r),\overline{y}_1(r)\right)$$
$$\overline{K}_2=h\overline{f}\left(t_1+\frac{h}{2},\underline{y}_1(r),\overline{y}_1(r)+\frac{\overline{K}_1}{2}\right)$$
$$\overline{K}_3=h\overline{f}\left(t_2+\frac{h}{2},\underline{y}_2(r),\overline{y}_2(r)+\frac{\overline{K}_2}{2}\right) \qquad Eq.12$$
$$\overline{K}_4=h\overline{f}\left(t_3+h,\underline{y}_3(r),\overline{y}_3(r)+h\right)$$
$$\overline{y}_{i+1}=\overline{y}_i+\frac{1}{6}\left(\overline{K}_1+2\overline{K}_2+2\overline{K}_3+\overline{K}_4\right)$$

Step 3
Let $n=3$

Step 4
Let $t_{n+1}=t_1+nh$

Step 5
Let the Milne-Simpson formula,

$$\underline{y}_{n+1}^{(P)}=\underline{y}_{n-3}+\frac{4}{3}h\left(2\underline{f}_n-\frac{4}{3}\underline{f}_{n-1}+2\underline{f}_{n-2}\right) \qquad \overline{y}_{n+1}^{(P)}=\overline{y}_{n-3}+\frac{4}{3}h\left(2\overline{f}_n-\frac{4}{3}\overline{f}_{n-1}+2\overline{f}_{n-2}\right)$$
$$\underline{y}_{n+1}^{(C)}=\underline{y}_{n-1}+\frac{h}{3}\left(\underline{f}_{n+1}-4\underline{f}_n+\underline{f}_{n-1}\right) \qquad \overline{y}_{n+1}^{(C)}=\overline{y}_{n-1}+\frac{h}{3}\left(\overline{f}_{n+1}-4\overline{f}_n+\overline{f}_{n-1}\right) \qquad Eq.13$$

Step 6
Algorithm is completed and $\left[\underline{y}(t_n;r),\overline{y}(t_n;r)\right]$ approximate to $\left[\underline{Y}(t_n;r),\overline{Y}(t_n;r)\right]$.

The results of the Milne-Simpson approach will be compared to the equivalent order Runge-Kutta method. The absolute error can be calculated by using formula

$$\left|y^{(i)}_{exact}-y^{(i)}_{approximate}\right|, \qquad Eq.14$$

where $y^{(i)}_{exact}$ is the value of exact solution and $y^{(i)}_{approximate}$ is the value of approximate solution which refer to the numerical solution by Milne-Simpson method and Runge-Kutta method.

3. Results and Discussion

This research focus on Milne-Simpson method to solve FDEs using Hukuhara approach where the solution is generated numerically using MATLAB. Two sets of first-order FDEs problems are numerically tested using constant step size of $h = 0.1$. The results will be compared with existing RK method of order four.

3.1 Test Problem (a) Source: Ma et al [12]

The FIVP

$$y'(t) = y(t) \qquad Eq.15$$

with the initial conditions

$$y(0) = (0.75+0.25r, 1.125-0.125r). \qquad Eq.16$$

Exact solution at $t = 1$ is given by

$$Y(1;r) = [(0.75+0.25r)e, (1.125-0.125r)e], \quad 0 \le r \le 1. \qquad Eq.17$$

Table 1 and 2 show the result of exact solution and approximate solution between Milne-Simpson (MS) and fourth order Runge-Kutta (RK4) method for Test Problem (a) using same step size. Absolute error between these two methods is obtained. The figures illustrate the graph for error of MS, and RK4 method.

Table 1: Numerical result of lower bound of MS and RK4 methods for solving Test Problem (a) for $h = 0.1$ at $t = 1$.

R	Exact Solution	Numerical Results		Absolute Error	
		MS	RK4	MS	RK4
0.0	2.0387113713	2.0387111371	2.0387098081	$2.34207e-07$	$1.56324e-06$
0.1	2.1066684171	2.1066681750	2.1066668017	$2.42014e-07$	$1.61535e-06$
0.2	2.1746254628	2.1746252129	2.1746237953	$2.49821e-07$	$1.66746e-06$
0.3	2.2425825085	2.2425822509	2.2425807889	$2.57628e-07$	$1.71957e-06$
0.4	2.3105395542	2.3105392888	2.3105377825	$2.65435e-07$	$1.77168e-06$
0.5	2.3784965999	2.3784963267	2.3784947761	$2.73242e-07$	$1.82378e-06$
0.6	2.4464536456	2.4464533646	2.4464517697	$2.81049e-07$	$1.87589e-06$
0.7	2.5144106913	2.5144104025	2.5144087633	$2.88855e-07$	$1.92800e-06$
0.8	2.5823677370	2.5823674404	2.5823657569	$2.96662e-07$	$1.98011e-06$
0.9	2.6503247827	2.6503244783	2.6503227505	$3.04469e-07$	$2.03222e-06$
1.0	2.7182818285	2.7182818285	2.7182797441	$3.12276e-07$	$2.05207e-06$

Table 2: Numerical result of upper bound of MS and RK4 methods for solving Test Problem (a) for $h = 0.1$ at $t = 1$.

R	Exact Solution	Numerical Results		Absolute Error	
		MS	RK4	MS	RK4
0.0	3.0580670570	3.0580667057	3.0580647122	$3.51311e-07$	$2.34486e-07$
0.1	3.0240885342	3.0240881868	3.0240862154	$3.47407e-07$	$2.31881e-06$
0.2	2.9901100113	2.9901096678	2.9901077185	$3.43504e-07$	$2.29276e-06$
0.3	2.9561314884	2.9561311488	2.9561292217	$3.39600e-07$	$2.26670e-06$
0.4	2.9221529656	2.9221526299	2.9221507249	$3.35697e-07$	$2.24065e-06$
0.5	2.8881744427	2.8881741109	2.8881722281	$3.31793e-07$	$2.21459e-06$
0.6	2.8541959199	2.8541955920	2.8541937313	$3.27890e-07$	$2.18854e-06$
0.7	2.8202173970	2.8202170730	2.8202152345	$3.23987e-07$	$2.16249e-06$
0.8	2.7862388742	2.7862385541	2.7862367377	$3.20083e-07$	$2.13643e-06$
0.9	2.7522603513	2.7522600351	2.7522582409	$3.16180e-07$	$2.11038e-06$

1.0	2.7182818285	2.7182815162	2.7182797441	3.12276*e* − 07	2.04207*e* − 06

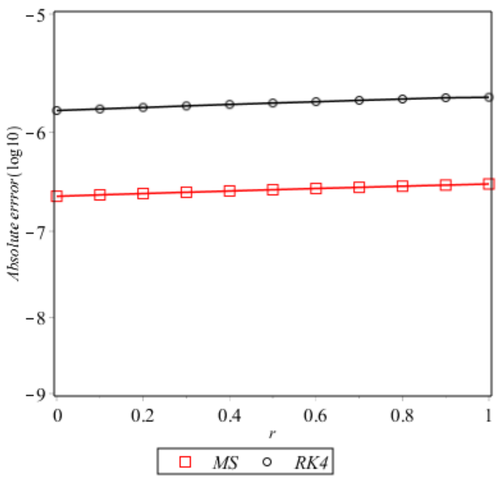

Figure 1: Error lower bound of MS and RK4 solution for Test Problem (a) for $h = 0.1$ at $t=1$

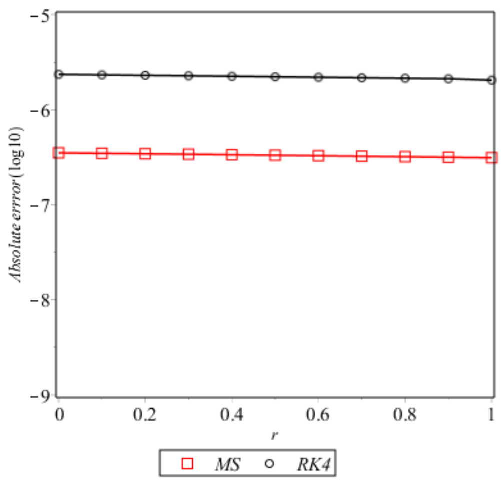

Figure 2: Error upper bound of MS and RK4 solution for Test Problem (a) for $h = 0.1$ at $t=1$

Table 1 shows the numerical results of lower bound of MS and RK4 and Table 2 represents the numerical results of upper bound of MS and RK4 for solving Test Problem (a) by using $h = 0.1$. The results are compared and it show that the results give a better accuracy for MS method than RK4 method. The absolute error show that the MS method gives a better accuracy as the absolute error is much smaller than RK4 method.

Figure 1 and 2 show the comparison between error of MS and RK4. The graphs obtained shown that MS and RK4 method has a difference numerical approximations. The results can be proven from Table 1 and 2.

3.2 Test Problem (b) Source: Ghazanfari [13]

The FIVP

$$y'(t)=ty(t), \qquad Eq.18$$

with the initial conditions

$$y(0)=(1.01+0.1r\sqrt{e},1.5+0.1r\sqrt{e}) \qquad Eq.19$$

Exact solution at $t=1$ is given by

$$Y(1;r)=\left[\left(1.01+0.1r\sqrt{e}\right)e^{\frac{1}{2}t^2},\left(1.5+0.1r\sqrt{e}\right)e^{\frac{1}{2}t^2}\right],\ \ 0\le r\le 1. \qquad Eq.20$$

Table 3 and 4 show the result of exact solution and approximate solution between MS and RK4 method for Test Problem (b) using same step size. Absolute error between these two methods is obtained. Figure 3 and 4 illustrates the graph for error of MS, and RK4 method.

Table 3: Numerical result of lower bound of MS and RK4 methods for solving Test Problem (b) for $h=0.1$ at $t=1$.

R	Exact Solution	Numerical Results		Absolute Error	
		MS	RK4	MS	RK4
0.0	1.6652084834	1.6652122016	1.6652082171	3.71821*e* − 06	2.66283*e* − 07
0.1	1.6923913017	1.6923950806	1.6923910311	3.77891*e* − 06	2.70630*e* − 07

0.2	1.7195741200	1.7195779596	1.7195738450	$3.83960e-06$	$2.74977e-07$
0.3	1.7467569383	1.7467608386	1.7467566589	$3.90030e-06$	$2.79324e-07$
0.4	1.7739397565	1.7739437175	1.7739394729	$3.96099e-06$	$2.83670e-07$
0.5	1.8011225748	1.8011265965	1.8011222868	$4.02169e-06$	$2.88017e-07$
0.6	1.8283053931	1.8283094755	1.8283051008	$4.08238e-06$	$2.92364e-07$
0.7	1.8554882114	1.8554923545	1.8554879147	$4.14308e-06$	$2.96711e-07$
0.8	1.8826710297	1.8826752335	1.8826707286	$4.20378e-06$	$3.01058e-07$
0.9	1.9098538480	1.9098581124	1.9098535426	$4.26447e-06$	$3.05404e-07$
1.0	1.9370366663	1.9370409914	1.9370363565	$4.32517e-06$	$3.09751e-07$

Table 4: Numerical result of upper bound of MS and RK4 methods for solving Test Problem (b) for $h=0.1$ at $t=1$.

R	Exact Solution	Numerical Results		Absolute Error	
		MS	RK4	MS	RK4
0.0	2.4730819061	2.4730874281	2.4730815106	$5.52209e-06$	3.95470e − 07
0.1	2.5002647243	2.5002703071	2.5002643245	$5.58279e-06$	3.99817e − 07
0.2	2.5274475426	2.5274531861	2.5274471385	$5.64348e-06$	4.04164e − 07
0.3	2.5546303609	2.5546360651	2.5546299524	$5.70418e-06$	4.08510e − 07
0.4	2.5818131792	2.5818189441	2.5818127663	$5.76488e-06$	4.12857e − 07
0.5	2.6089959975	2.6090018230	2.6089955803	$5.82557e-06$	4.17204e − 07
0.6	2.6361788158	2.6361847020	2.6361783942	$5.88627e-06$	4.21551e − 07
0.7	2.6633616340	2.6633675810	2.6633612081	$5.94696e-06$	4.25898e − 07
0.8	2.6905444523	2.6905504600	2.6905440221	$6.00766e-06$	4.30244e − 07
0.9	2.7177272706	2.7177333390	2.7177268360	$6.06836e-06$	4.34591e − 07
1.0	2.7449100889	2.7449162179	2.7449096500	$6.12905e-06$	4.38938e − 07

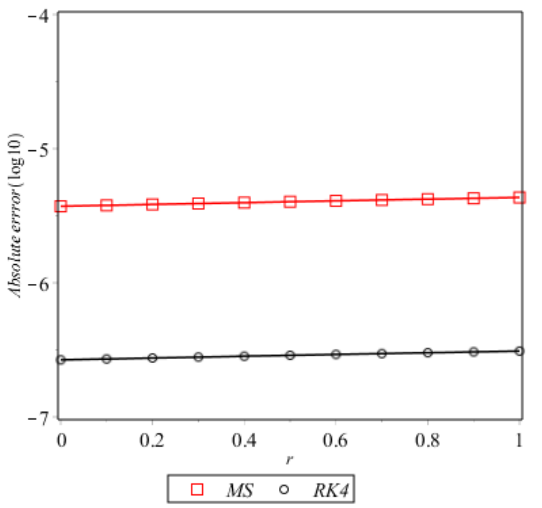

Figure 3: Error lower bound of MS and RK4 solution for Test Problem (b) for $h = 0.1$ at $t=1$

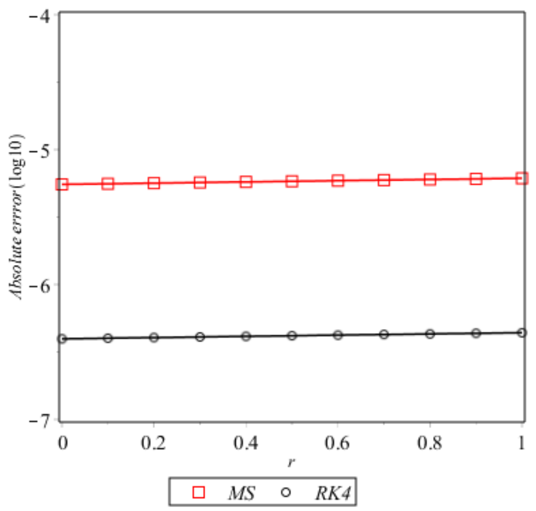

Figure 4: Error upper bound of MS and RK4 solution for Test Problem (b) for $h = 0.1$ at $t=1$

Table 3 shows the numerical results of lower bound of MS and RK4 and Table 4 represents the numerical results of lower bound of MS and RK4 for solving Test Problem (b) by using $h = 0.1$. The results are compared and it show that the results give a better accuracy for MS method than RK4 method. The absolute error shows that the RK4 give a better result as the absolute error is much smaller than MS.

Figure 3 and 4 show the comparison between error of MS and RK4. The graphs obtained by using Matlab 2018. As shown in the figure, MS and RK4 method has a differences numerical approximations. The results can be proven from Table 3 and 4.

4. Conclusion

In this paper, MS method is proposed. It can be said that the proposed method is suitable in solving first order FDEs. The numerical approximation of the solution is then compared with RK4 method and the result shows that MS method is comparable to RK4 method. The absolute error for MS method is smaller compared to RK4 method. It can be concluded that MS method is more accurate in solving first order FDEs using Hukuhara approach compared to RK4.

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