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On $\theta-G_{\mathcal{N}}-$Precontinuous Functions in Grill Topological Spaces

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Abstract : In this paper, we introduced and investigated the notions of $\theta-G_N-$preclosed sets in grill topological space, $\theta-G_N-$precontinuous function and strongly $\theta-G_N-$precontinuous function. Furthermore, we studied the relations between the $\theta-G_N-$precontinuous and other known continuous function. They will be introduced in grill topological spaces by using the G_N-preopen sets.

Keywords: Closed Sets, Continuous Function, Grill Topological Spaces.

1. Introduction

The idea of grill on a topological space, given by Choquet [2], goes as follows: A non-null collection G of subsets of a topological spaces X is said to be a grill on X if

(i) $A \in G$ and $A \subseteq B \Rightarrow B \in G$

(ii) $A,B \subseteq X$ and $A \cup B \in G \Rightarrow A \in G$ or $B \in G$.

For a topological space X, the operator $\Phi : P(X) \rightarrow P(X)$, given by[5]

$\Phi(A) = \{x \in X : U \cap A \in G$, for each open neighborhood U of $x\}$,

and the operator $\Psi: P(X) \rightarrow P(X)$, given by $\Psi(A)= A \cup \Phi(A)$. Then there exists a unique topology τ_G on X given by $\tau_G = \{U \subseteq X : \Psi(X-U) = X-U\}$, such that $\tau \subseteq \tau_G$.

In 1968 Velicko [3] introduced the notions of $\theta-$open sets. In 1982 Mashhour [4], introduced the notion of a precontinuous function. In 2009 Al-Omari and Noiri [1], introduced

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the notions of N−precontinuous function. In 2010 Hatir and Jafari [3], introduced the notions of G−precontinuous function.

For a topological space (X,τ) and $A\subseteq X$, throughout this paper, we mean $Cl(A)$ and $Int(A)$ the closure set and the interior set of A, respectively.

Definition 1.1. [9] Let (X,τ) be topological space and $A\subseteq X$. A point $x\in X$ is called $\theta-$cluster point of A if $Cl(U)\cap A\neq\phi$ for every open set U in X containing x.

The set of all $\theta-$cluster points of A is called the $\theta-$closure set of A and denoted by $Cl^{\theta}(A)$. A subset A of topological space (X,τ) is called $\theta-$closed set in (X,τ) if $Cl^{\theta}(A)=A$. The complement of $\theta-$closed set in (X,τ) is called $\theta-$open set.

Theorem 1.2. [9] Every $\theta-$closed set in topological space (X,τ) is closed set.

Definition 1.3. [9] A function $f:(X,\tau)\rightarrow(Y,\rho)$ of a topological space (X,τ) into a space (Y,ρ) is called $\theta-$continuous if for each $x\in X$ and each V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(Cl(U))\subseteq Cl(V)$.

Definition 1.4. [6] A subset A of a grill topological space (X,τ,G) is called a *$G\mathcal{N}-$preopen set* if for each $x\in A$, there exists a $G-$preopen set U_x containing x such that U_x-A is a finite set. The complement of $G\mathcal{N}-$preopen set is called $G\mathcal{N}-$preclosed set.

Theorem 1.5. [6] The intersection of an open set and $G\mathcal{N}-$preopen set is a $G\mathcal{N}-$preopen set.

Definition 1.6. [6] Let (X,τ,G) be a grill topological space and $A\subseteq X$. The *$G\mathcal{N}-$closure* set of A is defined as the intersection of all $G\mathcal{N}-$preclosed subsets of X containing A and is denoted by ${}_{G\mathcal{N}}Cl(A)$. The *$G\mathcal{N}-$interior* set of A is defined as the union of all $G\mathcal{N}-$preopen subsets of X contained in A and is denoted by ${}_{G\mathcal{N}}Int(A)$.

Theorem 1.7. [6] For a subset $A\subseteq X$ of grill topological space (X,τ,G), the following hold:

1. If U is an open set in X, then ${}_{G\mathcal{N}}Cl(A)\cap U\subseteq {}_{G\mathcal{N}}Cl(A\cap U)$.
2. If U is a closed set in X, then ${}_{G\mathcal{N}}Int(A\cup U)\subseteq {}_{G\mathcal{N}}Int(A)\cup U$.

Definition 1.8. [8] A function $f:(X,\tau,G)\rightarrow(Y,\rho)$ of a grill topological space (X,τ,G) into a topological space (Y,ρ) is called *$G\mathcal{N}-$precontinuous* if $f^{-1}(U)$ is a $G\mathcal{N}-$preopen set in (X,τ,G) for every open set U in Y.

Definition 1.9. [8] A function $f:(X,\tau,G)\rightarrow(Y,\rho)$ of a grill topological space (X,τ,G) into a space (Y,ρ) is called:

1. An *almost $G\mathcal{N}-$precontinuous* if for each $x\in X$ and each open set V in Y containing $f(x)$, there is a $G\mathcal{N}-$preopen set U in (X,τ,G) containing x such that $f(U)\subseteq Int[Cl(V)]$.

2. *Weakly $G\mathcal{N}-$precontinuous* function, if for each $x\in X$ and each open set V in Y containing $f(x)$, there is a $G\mathcal{N}-$preopen set U in (X,τ,G) containing x such that $f(U)\subseteq Cl(V)$.

The purpose of this paper is extend the notion of $G\mathcal{N}-$precontinuous by giving the concept of functions is called

θ- $G\mathcal{N}$-precontinuous in a grill topological space.

2. $\theta-GN-$Preclosed set

Let (X,τ,G) be grill topological space and $A\subseteq X$. A point $x\in X$ is called $\theta-G_{\mathcal{N}}-$precluster point of A if ${}_{G\mathcal{N}}Cl(U)\cap A\neq\phi$ for every $G_{\mathcal{N}}-$preopen set U containing x. The set of all $\theta-G_{\mathcal{N}}-$precluster points of A is called the *$\theta-G_{\mathcal{N}}-$preclosure set* of A and denoted by ${}_{G\mathcal{N}}Cl^{\theta}(A)$.

Definition 2.1. A subset A of grill topological space (X,τ,G) is called *$\theta-G_{\mathcal{N}}-$preclosed set*, if ${}_{G\mathcal{N}}Cl^{\theta}(A)=A$. The complement of $\theta-G_{\mathcal{N}}-$preclosed set is called $\theta-G_{\mathcal{N}}-$preopen set in (X,τ,G).

Theorem 2.2. Every $\theta-$closed set in topological space (X,τ) is $\theta-G_{\mathcal{N}}-$preclosed set in grill topological space (X,τ,G).

Proof. Let A be a $\theta-$closed set in a space (X,τ), that is, $Cl^{\theta}(A)=A$. It is clear that $A\subseteq {}_{G\mathcal{N}}Cl^{\theta}(A)$. We prove that ${}_{G\mathcal{N}}Cl^{\theta}(A)\subseteq A$. Let $x\in {}_{G\mathcal{N}}Cl^{\theta}(A)$. Then ${}_{G\mathcal{N}}Cl(U)\cap A\neq\phi$. Since ${}_{G\mathcal{N}}Cl(U)\subseteq Cl(U)$, then $Cl(U)\cap A\neq\phi$. Then $x\in Cl^{\theta}(A)=A$. Hence ${}_{G\mathcal{N}}Cl^{\theta}(A)\subseteq A$. That is, A is a $\theta-G_{\mathcal{N}}-$preclosed set in (X,τ,G).

The converse of the last theorem no need to be true.

Example 2.3. In a grill topological space (X,τ,G), where $X=\{a,b,c\}$, $\tau=\{\phi,X,\{a,b\}\}$ and $G=\{\{c\},\{a,c\},\{b,c\},X\}$, the set $\{a\}$ is a $\theta-G_{\mathcal{N}}-$preclosed set in (X,τ,G), but it is not $\theta-$closed set in (X,τ).

Theorem 2.4. Every $\theta-G_{\mathcal{N}}-$preclosed set is $G_{\mathcal{N}}-$preclosed set.

Proof. Let (X,τ,G) be a grill topological space and A be a $\theta-G_{\mathcal{N}}-$preclosed set, that is, ${}_{G\mathcal{N}}Cl^{\theta}(A)=A$. It is clear that $A\subseteq {}_{G\mathcal{N}}Cl(A)$. We prove that ${}_{G\mathcal{N}}Cl(A)\subseteq A$. Let $x\in {}_{G\mathcal{N}}Cl(A)$. Then $U\cap A\neq\phi$. Since $U\subseteq {}_{G\mathcal{N}}Cl(U)$, then ${}_{G\mathcal{N}}Cl(U)\cap A\neq\phi$. Then $x\in {}_{G\mathcal{N}}Cl^{\theta}(A)=A$. Hence ${}_{G\mathcal{N}}Cl(A)\subseteq A$. That is, A is a $G\mathcal{N}-$preclosed set in (X,τ,G).

Theorem 2.5. For open set H in grill topological space (X,τ,G), ${}_{G\mathcal{N}}Cl^{\theta}(H)={}_{G\mathcal{N}}Cl(H)$.

Proof. Let $x\in {}_{G\mathcal{N}}Cl(H)$. Then for every $G\mathcal{N}-$preopen set U in (X,τ,G) containing x, $U\cap H\neq\phi$. Since $U\subseteq {}_{G\mathcal{N}}Cl(U)$, then ${}_{G\mathcal{N}}Cl(U)\cap H\neq\phi$. Hence $x\in {}_{G\mathcal{N}}Cl^{\theta}(H)$. That is, ${}_{G\mathcal{N}}Cl(H)\subseteq {}_{G\mathcal{N}}Cl^{\theta}(H)$. For the other side, let $x\in {}_{G\mathcal{N}}Cl^{\theta}(H)$. Then for every $G\mathcal{N}-$preopen set U in (X,τ,G) containing x, ${}_{G\mathcal{N}}Cl(U)\cap H\neq\phi$. Since H is open set, then by Theorem (1.7), ${}_{G\mathcal{N}}Cl(U)\cap H\subseteq {}_{G\mathcal{N}}Cl(U\cap H)$. Then ${}_{G\mathcal{N}}Cl(U\cap H)\neq\phi$. Hence $U\cap H\neq\phi$. That is, $x\in {}_{G\mathcal{N}}Cl(H)$. That is, ${}_{G\mathcal{N}}Cl^{\theta}(H)\subseteq {}_{G\mathcal{N}}Cl(H)$.

Theorem 2.6. A subset U is $\theta-G_{\mathcal{N}}-$preopen set in grill topological space (X,τ,G) if and only if for each $x\in U$ there is $G_{\mathcal{N}}-$preopen set V in (X,τ,G) containing x such that ${}_{G\mathcal{N}}Cl(V)\subseteq U$.

Proof. Suppose that U is $\theta-G_{\mathcal{N}}-$preopen set in (X,τ,G) and $x\in U$. Then $x\notin X-U={}_{G\mathcal{N}}Cl^{\theta}(X-U)$. Then there is $G_{\mathcal{N}}-$preopen set V in (X,τ,G) containing x such that ${}_{G\mathcal{N}}Cl(V)\cap(X-U)=\phi$. That is, ${}_{G\mathcal{N}}Cl(V)\subseteq U$.

Conversely, suppose that U is not $\theta-G_{\mathcal{N}}-$preopen set. Then $X-U$ is not $\theta-G_{\mathcal{N}}-$preclosed set. That is, there is $x\in {}_{G\mathcal{N}}Cl^{\theta}(X-U)$ and $x\notin X-U$. Since $x\in U$, then by the hypothesis, there is $G_{\mathcal{N}}-$preopen set V in (X,τ,G) containing x such that ${}_{G\mathcal{N}}Cl(V)\subseteq U$. This implies, ${}_{G\mathcal{N}}Cl(V)\cap(X-U)=\phi$ and this contradiction. Hence U is $\theta-G_{\mathcal{N}}-$preopen set.

3. $\theta-GN-$Precontinuous Functions

Definition 3.1. A function $f:(X,\tau,G) \to (Y,\rho)$ of a grill topological space (X,τ,G) into a space (Y,ρ) is called *$\theta - G_{\mathcal{N}}$-precontinuous function* if for each $x \in X$ and each open set V in (Y,ρ) containing $f(x)$, there exists $G_{\mathcal{N}}$-preopen set U in (X,τ,G) containing x such that $f({}_{G_{\mathcal{N}}}Cl(U)) \subseteq Cl(V)$.

Theorem 3.2. A function $f:(X,\tau,G) \to (Y,\rho)$ is $\theta - G_{\mathcal{N}}$-precontinuous if and only if

$$ {}_{G_N}Cl^{\theta}(f^{-1}(V)) \subseteq f^{-1}(Cl(V)) $$

for every open set V in (Y,ρ).

Proof. Suppose that f is $\theta - G_{\mathcal{N}}$-precontinuous. Let V be any open set in (Y,ρ). Let $x \notin f^{-1}(Cl(V))$. Then $f(x) \notin Cl(V)$. Then $f(x) \in Y - Cl(V)$. Since $Y - Cl(V)$ is open set in (Y,ρ) and f is $\theta - G_{\mathcal{N}}$-precontinuous, then there exists $G_{\mathcal{N}}$-preopen set U in (X,τ,G) containing x such that

$$ f({}_{G_N}Cl(U)) \subseteq Cl(Y - Cl(V)). $$

This implies,

$$ f({}_{G_N}\mathrm{Cl(U)}) \subseteq \mathrm{Cl(Y - Cl(V))} = \mathrm{Y - Int(Cl(V))}. $$

Hence

$$ f({}_{G_N}\mathrm{Cl(U)}) \cap \mathrm{Int(Cl(V))} = \phi. $$

Since

$$ \mathrm{V} = \mathrm{Int(V)} \subseteq Int(Cl(V)), $$

then $f({}_{G_{\mathcal{N}}}Cl(U)) \cap V = \phi$ and so ${}_{G_{\mathcal{N}}}Cl(U) \cap f^{-1}(V) = \phi$. Since U is $G_{\mathcal{N}}$-preopen set, then $x \notin {}_{G_{\mathcal{N}}}Cl^{\theta}(f^{-1}(V))$. Hence

$$ {}_{G_N}Cl^{\theta}(f^{-1}(V)) \subseteq f^{-1}(Cl(V)) $$

Conversely, let $x \in X$ be any point and V be any open set in (Y,ρ) containing $f(x)$. Since $V \cap (Y - Cl(V)) = \phi$, then $f(x) \notin Cl(Y - Cl(V))$. This implies, $x \notin f^{-1}[Cl(Y - Cl(V))]$. Since $Y - Cl(V)$ is an open set in (Y,ρ), then by the hypothesis,

$$ {}_{G_N}Cl^{\theta}[f^{-1}(\mathrm{Y - Cl(V)})] \subseteq f^{-1}[\mathrm{C}l(\mathrm{Y - Cl(V)})] $$

Then $x \notin {}_{G_{\mathcal{N}}}Cl^{\theta}[f^{-1}(Y - Cl(V))]$. Hence there is $G_{\mathcal{N}}$-preopen set U in (X,τ,G) containing x such that

$$ {}_{G_N}\mathrm{C}l(\mathrm{U}) \cap f^{-1}(\mathrm{Y - C}l(\mathrm{V})) = \phi. $$

This implies, $f({}_{G_{\mathcal{N}}}Cl(U)) \subseteq Cl(V)$. Hence f is $\theta - G_{\mathcal{N}}$-precontinuous.

Theorem 3.3. A function $f:(X,\tau,G) \to (Y,\rho)$ is $\theta - G_{\mathcal{N}}$-precontinuous if and only if

$$ {}_{G_N}\mathrm{Cl}^{\theta}[X - f^{-1}(\mathrm{C}l(\mathrm{V}))] \subseteq X - f^{-1}(\mathrm{V}), $$

for every open set V in (Y,ρ).

Proof. Suppose that f is $\theta-G_{\mathcal{N}}-$precontinuous. Let V be any open set in (Y,ρ). Let $x\notin X-f^{-1}(V)$. Then $f(x)\in V$. Since f is $\theta-G_{\mathcal{N}}-$precontinuous, then there exists $G_{\mathcal{N}}-$preopen set U in (X,τ,G) containing x such that $f({}_{G_\mathcal{N}}Cl(U))\subseteq Cl(V)$. This implies, ${}_{G_\mathcal{N}}Cl(U)\subseteq f^{-1}(Cl(V))$. Then

$$_{G_N}Cl(U)\cap[X-f^{-1}(Cl(V))]=\phi..$$

Since U is a $G_{\mathcal{N}}-$preopen set, then $x\notin {}_{G_\mathcal{N}}Cl^{\theta}[X-f^{-1}(Cl(V))]$. Hence

$$_{G_N}Cl^{\theta}[X-f^{-1}(Cl(V))]\subseteq X-f^{-1}(V).$$

Conversely, let $x\in X$ be any point and V be any open set in (Y,ρ) containing $f(x)$. Then $x\in f^{-1}(V)$, that is, $x\notin X-f^{-1}(V)$. Then by the hypothesis, $x\notin {}_{G_\mathcal{N}}Cl^{\theta}[X-f^{-1}(Cl(V))]$. That is, there is $G_{\mathcal{N}}-$preopen set U in (X,τ,G) containing x such that

$$_{G_N}Cl(U)\cap[X-f^{-1}(Cl(V))]=\phi.$$

This implies, ${}_{G_\mathcal{N}}Cl(U)\subseteq f^{-1}(Cl(V))$ and so $f({}_{G_\mathcal{N}}Cl(U))\subseteq Cl(V)$. Hence f is $\theta-G_{\mathcal{N}}-$precontinuous.

Theorem 3.4. For a function $f:(X,\tau,G)\rightarrow(Y,\rho)$, the following properties are equivalent:

1. f is $\theta-G_{\mathcal{N}}-$precontinuous.
2. ${}_{G_\mathcal{N}}Cl^{\theta}(f^{-1}(B))\subseteq f^{-1}(Cl^{\theta}(B))$ for every subset $B\subseteq Y$.
3. $f({}_{G_\mathcal{N}}Cl^{\theta}(A))\subseteq Cl^{\theta}(f(A))$ for every subset $A\subseteq X$.

Proof. (1) ⇒ (2): Let B be any subset of Y. Suppose that $x\notin f^{-1}(Cl^{\theta}(B))$. Then $f(x)\notin Cl^{\theta}(B)$. Then there is an open set V in Y containing $f(x)$ such that $Cl(V)\cap B=\phi$. Since f is $\theta-G_{\mathcal{N}}-$precontinuous, then there exists $G_{\mathcal{N}}-$preopen set U in (X,τ,G) containing x such that $f({}_{G_\mathcal{N}}Cl(U))\subseteq Cl(V)$. Then we have $f({}_{G_\mathcal{N}}Cl(U))\cap B=\phi$. This implies, ${}_{G_\mathcal{N}}Cl(U)\cap f^{-1}(B)=\phi$. Hence $x\notin {}_{G_\mathcal{N}}Cl^{\theta}(f^{-1}(B))$. That is,

$$_{G_N}Cl^{\theta}(f^{-1}(B))\subseteq f^{-1}(Cl^{\theta}(B)).$$

(2) ⇒ (1): Let $x\in X$ and V be any open set in (Y,ρ) containing $f(x)$. Since $Cl(V)\cap(Y-Cl(V))=\phi$, then $f(x)\notin Cl^{\theta}(Y-Cl(V))$. This implies, $x\notin f^{-1}[Cl^{\theta}(Y-Cl(V))]$. Since $Y-Cl(V)\subseteq Y$, then by the hypothesis,

$$_{G_N}Cl^{\theta}[f^{-1}(Y-Cl(V))]\subseteq f^{-1}(Cl^{\theta}(Y-Cl(V)).$$

Then $x\notin {}_{G_\mathcal{N}}Cl^{\theta}[f^{-1}(Y-Cl(V))]$. Hence there is $G_{\mathcal{N}}-$preopen set U in (X,τ,G) containing x such that

$$_{G_N}Cl(U)\cap f^{-1}(Y-Cl(V))=\phi.$$

This implies, $f({}_{G_\mathcal{N}}Cl(U))\subseteq Cl(V)$. Hence f is $\theta-G_{\mathcal{N}}-$precontinuous.

(2) ⇒ (3): Let A be any subset of X. Since $f(A)\subseteq Y$, then by the hypothesis,

$$_{G_N}Cl^{\theta}(A)\subseteq{}_{G_N}Cl^{\theta}[f^{-1}(f(A))]\subseteq f^{-1}[Cl^{\theta}(f(A))].$$

This implies, $f({}_{G_\mathcal{N}}Cl^{\theta}(A))\subseteq Cl^{\theta}(f(A))$.

(3) ⇒ (2): Let B be any subset of Y. Since $f^{-1}(B)\subseteq X$, then by the hypothesis,

$$f[{}_{G_N}Cl^{\theta}(f^{-1}(B))]\subseteq Cl^{\theta}[f(f^{-1}(B))]\subseteq Cl^{\theta}(B).$$

This implies, ${}_{G\mathcal{N}}Cl^{\theta}(f^{-1}(B))\subseteq f^{-1}(Cl^{\theta}(B))$.

Lemma 3.5. Let $f:(X,\tau,G)\rightarrow(Y,\rho)$ be a function. Then the following statements are equivalent:

1. f is an almost $G_{\mathcal{N}}-$precontinuous.
2. $f^{-1}(F)$ is $G_{\mathcal{N}}-$preclosed set in (X,τ,G) for every $r-$closed set F in Y.
3. $f^{-1}(V)$ is $G_{\mathcal{N}}-$preopen set in (X,τ,G) for every $r-$open set V in Y.

Proof. (1)⇒ (2): Let F be any $r-$closed set in (Y,ρ) and

$$x\in X-f^{-1}(F)=f^{-1}(Y-F).$$

Then $Y-F$ is $r-$open and, so open set in (Y,ρ) containing $f(x)$. Since f is an almost $G_{\mathcal{N}}-$precontinuous, then there is a $G_{\mathcal{N}}-$preopen set U in (X,τ,G) containing x such that

$$f(U)\subseteq Int[Cl(Y-F)]=Y-F.$$

This implies,

$$x\in U\subseteq f^{-1}(Y-F)=X-f^{-1}(F),$$

that is, $X-f^{-1}(F)$ is $G_{\mathcal{N}}-$preopen set. Hence $f^{-1}(F)$ is $G_{\mathcal{N}}-$preclosed set in (X,τ,G).

(2) ⇒ (3): It is trivial.

(3) ⇒ (1): Let $x\in X$ and V be any open set in (Y,ρ) containing $f(x)$. Since $Int(Cl(V))$ is $r-$open set in Y containing $f(x)$. Then by the hypothesis, $U=f^{-1}[Int(Cl(V))]$ is $G_{\mathcal{N}}-$preopen set in (X,τ,G) containing x and

$$f(U)=f[f^{-1}[Int(Cl(V))]]\subseteq Int(Cl(V)).$$

Hence f is an almost $G_{\mathcal{N}}-$precontinuous.

Theorem 3.6. Every almost $G_{\mathcal{N}}-$precontinuous is $\theta-G_{\mathcal{N}}-$precontinuous.

Proof. Let $f:(X,\tau,G)\rightarrow(Y,\rho)$ be almost $G_{\mathcal{N}}-$precontinuous. Let $x\in X$ be any point and V be any open set in (Y,ρ) containing $f(x)$. Since

$$Cl(V)=Cl[Int(V)]\subseteq Cl[Int(Cl(V))]$$

and

$$Cl[Int(Cl(V))]\subseteq Cl[Cl(V)]=Cl(V).$$

then $Cl(V)$ is $r-$closed set in Y. Since $Int(Cl(V))$ is $r-$open set in Y containing $f(x)$ and f is almost $G_{\mathcal{N}}-$precontinuous, then by Lemma (3.5), $U=f^{-1}[Int(Cl(V))]$ is $G_{\mathcal{N}}-$preopen set and $f^{-1}(Cl(V))$ is $G_{\mathcal{N}}-$preclosed set in (X,τ,G) and

$${}_{G_N}Cl(U)={}_{G_N}Cl[f^{-1}[Int(Cl(V))]]\subseteq{}_{G_N}Cl[f^{-1}(Cl(V))]=f^{-1}(Cl(V)).$$

This implies,

$$f({}_{G_N}Cl(U)) \subseteq Cl(V).$$

Hence f is $\theta - G_{\mathcal{N}}-$precontinuous.

Theorem 3.7. Every $\theta - G_{\mathcal{N}}-$precontinuous is weakly $G_{\mathcal{N}}-$precontinuous.

Proof. Let $f:(X,\tau,G) \to (Y,\rho)$ be $\theta - G_{\mathcal{N}}-$precontinuous. Let $x \in X$ be any point and V be any open set in (Y,ρ) containing $f(x)$. Then there exists $G_{\mathcal{N}}-$preopen set U in (X,τ,G) containing x such that $f({}_{G_{\mathcal{N}}}Cl(U)) \subseteq Cl(V)$. Since

$$f(U) \subseteq f({}_{G_N}Cl(U)) \subseteq Cl(V),$$

then f is weakly $G_{\mathcal{N}}-$precontinuous.

Definition 3.8. A function $f:(X,\tau,G) \to (Y,\rho)$ is called *strongly $\theta - G_{\mathcal{N}}-$precontinuous function* if for each $x \in X$ and each open set V in (Y,ρ) containing $f(x)$, there is $G_{\mathcal{N}}-$preopen set U in (X,τ,G) containing x such that $f({}_{G_{\mathcal{N}}}Cl(U)) \subseteq V$.

Theorem 3.9. A function $f:(X,\tau,G) \to (Y,\rho)$ is strongly $\theta - G_{\mathcal{N}}-$precontinuous if and only if $f^{-1}(V)$ is $\theta - G_{\mathcal{N}}-$preopen set in (X,τ,G) for every open set V in (Y,ρ).

Proof. Suppose that f is strongly $\theta - G_{\mathcal{N}}-$precontinuous. Let V be any open set in (Y,ρ). We prove that $X - f^{-1}(V)$ is $\theta - G_{\mathcal{N}}-$preclosed set. Let $x \notin X - f^{-1}(V)$. Then $f(x) \in V$. Since f is strongly $\theta - G_{\mathcal{N}}-$precontinuous, then there exists $G_{\mathcal{N}}-$preopen set U in (X,τ,G) containing x, such that $f({}_{G_{\mathcal{N}}}Cl(U)) \subseteq V$. This implies, ${}_{G_{\mathcal{N}}}Cl(U) \subseteq f^{-1}(V)$. Hence

$${}_{G_N}Cl(U) \cap X - f^{-1}(V) = \phi.$$

Since U is $G_{\mathcal{N}}-$preopen, then $x \notin {}_{G_{\mathcal{N}}}Cl^{\theta}(X - f^{-1}(V))$. Hence

$${}_{G_N}Cl^{\theta}(X - f^{-1}(V)) \subseteq X - f^{-1}(V).$$

Hence $f^{-1}(V)$ is $\theta - G_{\mathcal{N}}-$preopen set.

Conversely, let x be any point in X and V be any open set in (Y,ρ) containing $f(x)$. Then by the hypothesis, $f^{-1}(V)$ is $\theta - G_{\mathcal{N}}-$preopen set, that is, $X - f^{-1}(V)$ is $\theta - G_{\mathcal{N}}-$preclosed set. Since

$$x \notin X - f^{-1}(V) = {}_{G_N}Cl^{\theta}(X - f^{-1}(V)).$$

Then there is $G_{\mathcal{N}}-$preopen set U in (X,τ,G) containing x such that

$${}_{G_N}Cl(U) \cap X - f^{-1}(V) = \phi.$$

This implies, $f({}_{G_{\mathcal{N}}}Cl(U)) \subseteq V$. Hence f is strongly $\theta - G_{\mathcal{N}}-$precontinuous.

Corollary 3.10. A function $f:(X,\tau,G) \to (Y,\rho)$ is strongly $\theta - G_{\mathcal{N}}-$precontinuous if and only if $f^{-1}(V)$ is $\theta - G_{\mathcal{N}}-$preclosed set in (X,τ,G) for every closed set V in (Y,ρ).

Theorem 3.11. For a function $f:(X,\tau,G) \to (Y,\rho)$, the following properties are equivalent:

1. f is strongly $\theta - G_{\mathcal{N}}-$precontinuous.
2. $f({}_{G_{\mathcal{N}}}Cl^{\theta}(A)) \subseteq Cl(f(A))$ for every subset $A \subseteq X$.

3. ${}_{G_\mathcal{N}}Cl^{\theta}(f^{-1}(B))\subseteq f^{-1}(Cl(B))$ for every subset $B\subseteq Y$.

Proof. (1) ⇒ (2): Let A be any subset of X. Suppose that $y\notin Cl(f(A))$. Then there is an open set V in Y containing y such that $f(x)=y$ and $V\cap f(A)=\phi$. Since f is strongly $\theta-G_\mathcal{N}-$precontinuous, then there exists $G_\mathcal{N}-$preopen set U in (X,τ,G) containing x such that $f({}_{G_\mathcal{N}}Cl(U))\subseteq V$. Then we have

$$f[{}_{G_N}Cl(U)\cap A]\subseteq f({}_{G_N}Cl(U))\cap f(A)=\phi.$$

This implies, ${}_{G_\mathcal{N}}Cl(U)\cap A=\phi$. Hence $x\notin {}_{G_\mathcal{N}}Cl^{\theta}(A)$. That is, $y\notin f({}_{G_\mathcal{N}}Cl^{\theta}(A))$. Hence

$$f({}_{G_N}Cl^{\theta}(A))\subseteq Cl(f(A)).$$

(2) ⇒ (3): Let B be any subset of Y. Since $f^{-1}(B)\subseteq X$, then by the hypothesis,

$$f[{}_{G_N}Cl^{\theta}(f^{-1}(B))]\subseteq Cl[f(f^{-1}(B))]\subseteq Cl(B).$$

Hence

$${}_{G_N}Cl^{\theta}(f^{-1}(B))\subseteq f^{-1}(Cl(B)).$$

(3) ⇒ (1): Let V be any open set in (Y,ρ). Since $Y-V$ is closed set in Y and by the hypothesis,

$$\begin{aligned}{}_{G_N}Cl^{\theta}(X-f^{-1}(V)) &= {}_{G_N}Cl^{\theta}(f^{-1}(Y-V))\subseteq f^{-1}(Cl(Y-V))\\ &= f^{-1}(Y-V)=X-f^{-1}(V)\end{aligned}$$

Hence $X-f^{-1}(V)$ is $\theta-G_\mathcal{N}-$preclosed set. That is, $f^{-1}(V)$ is $\theta-G_\mathcal{N}-$preopen set. Then by Theorem (3.9), f is strongly $\theta-G_\mathcal{N}-$precontinuous.

Theorem 3.12. Every strongly $\theta-G_\mathcal{N}-$precontinuous is $G_\mathcal{N}-$precontinuous.

Proof. From Theorem (3.9) and the fact every $\theta-G_\mathcal{N}-$preopen set is $G_\mathcal{N}-$preopen set.

Theorem 3.13. Let (Y,ρ) be a regular space. Then, for a function $f:(X,\tau,G)\rightarrow(Y,\rho)$, the following properties are equivalent:

1. f is weakly $G_\mathcal{N}-$precontinuous.

2. f is $G_\mathcal{N}-$precontinuous.

3. f is strongly $\theta-G_\mathcal{N}-$precontinuous.

Proof. (1) ⇒ (2): Let f be weakly $G_\mathcal{N}-$precontinuous. Let x be any point in X and V be any open set in Y containing $f(x)$. Since Y is regular, then there is an open set M in Y containing $f(x)$ such that $Cl(M)\subseteq V$. Since f is weakly $G_\mathcal{N}-$precontinuous, then there is $G_\mathcal{N}-$preopen set U in (X,τ,G) containing x such that $f(U)\subseteq Cl(M)\subseteq V$. Hence f is $G_\mathcal{N}-$precontinuous.

(2) ⇒ (3): Let f be $G_\mathcal{N}-$precontinuous. Let $x\in X$ be any point and V be any open set in Y containing $f(x)$. Since Y is regular, then there is an open set M in Y containing $f(x)$ such that $Cl(M)\subseteq V$. Since $f^{-1}(M)$ is $G_\mathcal{N}-$preopen set and $f^{-1}(Cl(M))$ is $G_\mathcal{N}-$preclosed set in (X,τ,G). Let $U=f^{-1}(M)$. Then we have

$${}_{G_N}Cl(U)={}_{G_N}Cl(f^{-1}(M))\subseteq{}_{G_N}Cl(f^{-1}(Cl(M)))=f^{-1}(Cl(M)).$$

This implies,

$$f({}_{G_N}Cl(U)) \subseteq Cl(M) \subseteq V.$$

Hence f is strongly $\theta - G_{\mathcal{N}} -$precontinuous.

(3) ⇒ (1): The proof follows immediately from the definitions.

Corollary 3.14. Let *(Y,ρ)* be a regular space. Then, for a function $f:(X,\tau,G) \rightarrow (Y,\rho)$, the following properties are equivalent:

1. f is $\theta - G_{\mathcal{N}} -$precontinuous.
2. f is an almost $G_{\mathcal{N}} -$precontinuous.
3. f is weakly $G_{\mathcal{N}} -$precontinuous.
4. f is $G_{\mathcal{N}} -$precontinuous.
5. f is strongly $\theta - G_{\mathcal{N}} -$precontinuous.

4. Conclusion

The applications of $G_{\mathcal{N}} -$precontinuous functions is verey important in the area of mathematics, computer sciences and other areas. The notions in this article have been developed for the last notions in a grill topological space by giving a new concept. Moreover, they will play a significant role in solving some mathematical problems. We suggest to study notions of $\theta - G_{\mathcal{N}} -$disconnected sets, $\theta - G_{\mathcal{N}} -$connected, $\theta - G_{\mathcal{N}} -$precompact sets, and separation axioms.

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